Z-Transform

1- Basic Definition of the Z-Transform:

The **z-transform** of a **function** $x(n)$ is defined as**:**

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

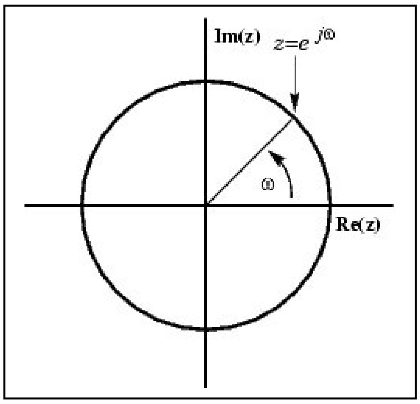
The power series for the z-transform is called **a Laurent series:**

So we can write that $X(z) = Ƶ\{x(n)\}$

There is a close relationship between the **z-transform** and the **Fourier transform** of a **discrete-time response** $h(n)$, which is defined as

$$H(e^{jw}) = \sum_{n=-\infty}^{\infty} h(n) e^{-jnw}$$

$$\boxed{z = e^{j\omega}}$$



The z-plane is a complex plane with an imaginary and real axis referring to the complex-valued variable z.

$$\therefore H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

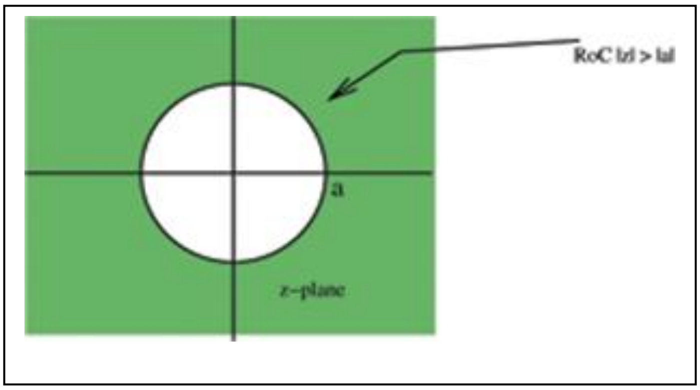
2- Region of Convergence:

The **ROC** for a given $x(n)$, is defined as the range of z for which the z-transform converges.

Example- 1: Find z-transform of $x(n) = a^n u(n)$ ***for* 0 < *a* < 1 ?**

__Solution:__ The z-transform is given by

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$



Which converges to

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a} \quad for$$

$$\left|az^{-1}\right| < |1| or\, |z| > |a|$$

Next: Another ROC example

$$z = x + iy, |z| = \sqrt{x^2 + y^2}, |z| > a,$$
$$\sqrt{x^2 + y^2} > a,$$
$$x^2 + y^2 > a^2,$$
$$x^2 + y^2 = a^2$$

Example- 2: Find z-transform of $x(n) = -b^{n} u(-n-1)$

<u>Solution:</u> The z-transform is given by

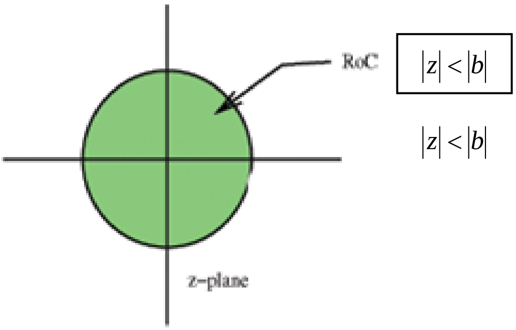
$$X(z) = \sum_{n=-\infty}^{\infty} x^{n}(n) z^{-n} = \sum_{n=-\infty}^{-1} -b^{n} z^{-n} \quad \text{or} \quad X(z) = -\sum_{n=1}^{\infty} b^{-n} z^{n}$$

$$X(z) = -\sum_{n=1}^{\infty} b^{-n} z^{n} \quad = -\sum_{n=1}^{\infty} \left[\frac{z}{b}\right]^{n} \quad = 1 - \sum_{n=0}^{\infty} \left[\frac{z}{b}\right]^{n}$$

The ROC in this case is the range of values where

$$X(z) = 1 - \frac{1}{1-b^{-1}z} = \frac{z}{z-b} \quad for$$

$$\left|b^{-1}z\right| < 1 or \left|z\right| < \left|b\right|$$



Example- 3: $x(n) = a^{n}u(n) - b^{n} u(-n-1)$

<u>Solution:</u> Using the results of Examples 1 and 2,

$$X(z) = \frac{z}{z-a} - \frac{z}{z-b}$$

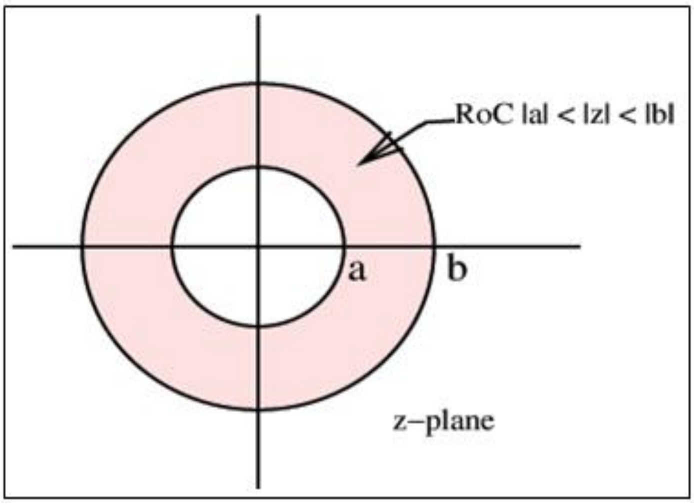
$$ROC1 \; |z| > a$$

$$ROC2 |z| < |b|$$

$$ROCt = ROC1 \cap ROC2$$

-Common area exist
for $\mathbf{b}^{\mathbf{n}} > a$ & $a = b$ (ROC rings)

and *for* $b < a$ (no ROC).



The ROC for x(n) is the intersection of the circle $z = be^{i\omega}$ and the circle $z = ae^{i\omega}$ as shown in Figure

3- Z-Transform properties:

- Linearity: if $x_1(n) \Longleftrightarrow X_1(z)$

  and $x_2(n) \Longleftrightarrow X_2(z)$

  Then for a_1 & a_2 constants

  $a_1 x_1(n) + a_2 x_2(n) \Longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$;

  $ROC_t = ROC_{x1} \cap ROC_{x2}$

- Uniqueness: if $x_1(n) \Longleftrightarrow X_1(z)$

  and $x_2(n) \Longleftrightarrow X_2(z)$

  if $x_1(n) \neq x_2(n)$ then $X_1(z) \neq X_2(z)$
- Time Shifting: if $x(n) \Longleftrightarrow X(z)$

  Then $x(n - n_0) \Longleftrightarrow z^{-n_0} X(z)$;
- Multiplication by an Exponential Sequence:

  if $x(n) \Longleftrightarrow X(z)$

  Then $a^n x(n) \Longleftrightarrow X(z)$ with $z \rightarrow z/a$;

  ROC= $|a|$. ROCx , so if $r_L <$ ROCx $< r_U$

  $|a|. r_L < |a|$ ROCx $< |a| r_U$

  $|a|. r_L <$ ROC $< |a| r_U$
- Multiplication by ramp:

  if $x(n) \Longleftrightarrow X(z)$

  Then $n\, x(n) \Longleftrightarrow - z \{d X(z) / dz\}$;

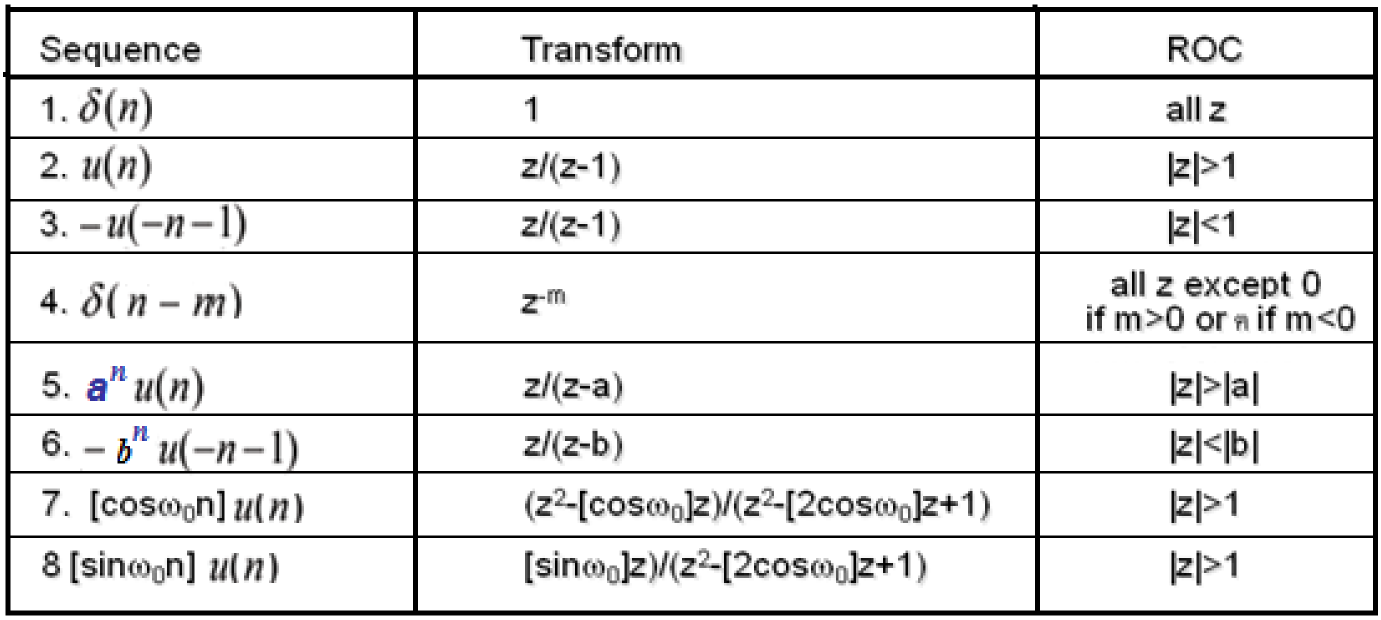
  ROC: ROCx except for the possible addition or deletion of the origin or infinity.
- Time Reversal:

  if $x(n) \Longleftrightarrow X(z)$

  Then $x(-n) \Longleftrightarrow X(1/z)$;

  ROC: 1/ROCx

4- Some Common z-Transform Pairs:



Sequence	Transform	ROC
1. $\delta(n)$	1	all z
2. $u(n)$	z/(z-1)	$\|z\|>1$
3. $-u(-n-1)$	z/(z-1)	$\|z\|<1$
4. $\delta(n-m)$	z^{-m}	all z except 0 if m>0 or n if m<0
5. $a^n u(n)$	z/(z-a)	$\|z\|>\|a\|$
6. $-b^n u(-n-1)$	z/(z-b)	$\|z\|<\|b\|$
7. $[\cos\omega_0 n]\, u(n)$	$(z^2-[\cos\omega_0]z)/(z^2-[2\cos\omega_0]z+1)$	$\|z\|>1$
8 $[\sin\omega_0 n]\, u(n)$	$[\sin\omega_0]z)/(z^2-[2\cos\omega_0]z+1)$	$\|z\|>1$

Example- 4: Find z-transform of

$$x(n) = (n-2)a^{n-2}\cos[\omega_o(n-2)]\ u(n-2)$$

Solution

$$X(z) = Z^{-2}\ [\overset{\top}{Z}\{\, n\cos[\omega_o(n)]\ u(n)\}]|z \longrightarrow z/a$$

for $|Z| > a$

$$= Z^{-2}\left[-Z\ \frac{d}{dz}\overset{\top}{Z}\{\cos[\omega_o(n)]\ u(n)\}\right]|z \longrightarrow z/a$$

$$= -Z^{-1}\left[\ \frac{d}{dz}\ \overset{\top}{Z}\{\cos[\omega_o(n)]\ u(n)\}\right]\ |z \longrightarrow z/a$$

$$= -Z^{-1}\left[\frac{d}{dz}\left(\frac{-Z^2\cos\omega_o + 2Z - \cos\omega_o}{(Z^2 - 2Z\cos\omega_o + 1)}\right)\right]|z \longrightarrow z/a$$

$$= -Z^{-1}\left[\frac{d}{dz}\left(\frac{-\left({}^{z}/_{a}\right)^{2}\cos\omega_o + 2\left({}^{z}/_{a}\right) - \cos\omega_o}{\left(\left({}^{z}/_{a}\right)^{2} - 2\left({}^{z}/_{a}\right)\cos\omega_o + 1\right)}\right)\right]$$

ROC=?

H.W.-1:

$$x(n) = (n-2)a^{n-2}\cos[\omega_o(n-2)]\ u(n-1)$$

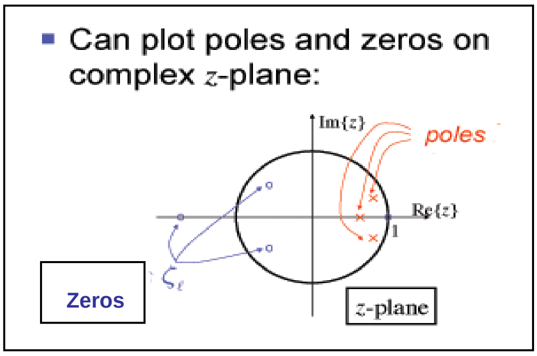
H.W.-2:

$$x(n) = n\,a^{n-3}\cos[\omega_o(n-2.5)]\ u(n-2)$$

5- Poles and Zeros

When *H*(*z*) is a rational function, i.e., a ration of polynomials in *z*, then:

- **The roots of the numerator polynomial are referred to as the *zeros* of *H*(*z*), and**
- **The roots of the denominator polynomial are referred to as the *poles* of *H*(*z*).**



- Can plot poles and zeros on complex z-plane:

Example- 5: For the response

$$h(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$$

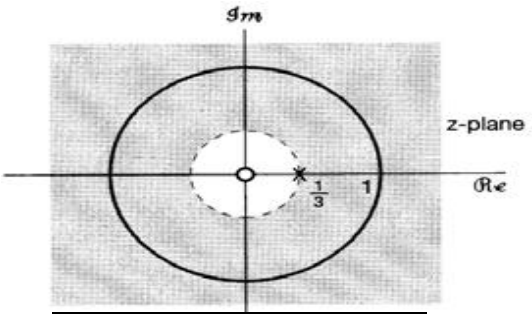
$$H(z) = \sum_{n=-\infty}^{+\infty} \left\{ 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n) \right\} z^{-n}$$

$$= 7 \sum_{n=-\infty}^{+\infty} \left(\frac{1}{3}\right)^n u(n)\, z^{-n} - 6 \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u(n)\, z^{-n}$$

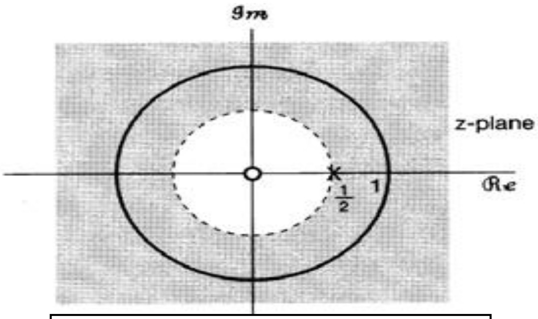
$$= 7\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n - 6\sum_{n=0}^{+\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

$$= \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$$

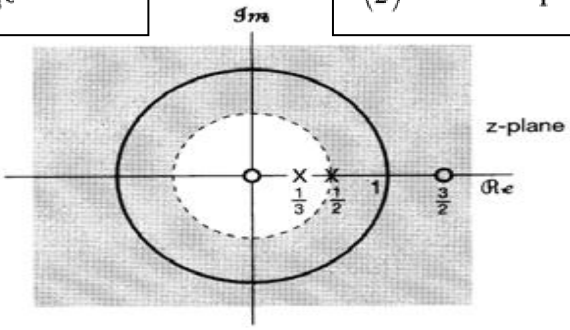
$$= \frac{z(z-\frac{3}{2})}{(z-\frac{1}{3})(z-\frac{1}{2})}.$$



$$\boxed{\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{3}z^{-1}}, \quad |z| > \frac{1}{3}}$$



$$\boxed{\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{1}{1-\frac{1}{2}z^{-1}}, \quad |z| > \frac{1}{2}}$$



(c)

$$\boxed{7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n) \xleftrightarrow{Z} \left(\frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}}\right), \quad |z| > \frac{1}{2}}$$

6- The Inverse Z-Transform

- Less formal ways sufficient most of the time
  - Inspection Method
  - Partial Fraction Expansion
  - Power Series Expansion
  - Inspection Method
    - Make use of known z-transform pairs such as

$$a^n u(n) \xleftrightarrow{Z} \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| > |a|$$

- Inverse Z-Transform by Partial Fraction Expansion:
- Assume that a given z-transform can be expressed as

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- Apply partial fractional expansion

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1-d_i z^{-1}\right)^m}$$

- First term exist only if $M > N$
  - B_r is obtained by long division
- Second term represents all first order poles
- Third term represents an order s pole
  - There will be a similar term for every high-order pole
- Each term can be inverse transformed by inspection

$$X(z)=\sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1,k\neq i}^{N} \frac{A_k}{1-d_k z^{-1}} + \sum_{m=1}^{s}\frac{C_m}{\left(1-d_i z^{-1}\right)^m}$$

- Coefficients are given as $A_k=\left(1-d_k z^{-1}\right)X(z)\Big|_{z=d_k}$

$$C_m=\frac{1}{(s-m)!(-d_i)^{s-m}}\left\{\frac{d^{s-m}}{dw^{s-m}}\left[\left(1-d_i w\right)^s X\left(w^{-1}\right)\right]\right\}_{w=d_i^{-1}}$$

Easier to understand with the following examples:

Example- 6: Find the inverse Z-Transform of

$$X(z)=\frac{Z}{3Z^2-4Z+1} \qquad \text{for ROC: a) } |z|>1, b)|Z|<\frac{1}{3}, c)\frac{1}{3}<|Z|<1$$

<u>Solution</u>

$$\frac{X(Z)}{Z}=\frac{1}{3Z^2-4Z+1}=\frac{1/3}{(Z-1/3)(Z-1)}=F(Z)$$

$$F(Z)=\frac{A}{(Z-1/3)}+\frac{B}{(Z-1)}$$

$$A=\lim F(Z)(Z-1/3)\to=\frac{1}{3(Z-1)}=-\frac{1}{2}$$

$$B=\lim F(Z)(Z-1)\to=\frac{1}{3(Z-1/3)}=\frac{1}{2}$$

$F(Z)=\dfrac{-1/2}{(Z-1/3)}+\dfrac{1/2}{(Z-1)}$,So $X(Z)=\dfrac{-(1/2)Z}{(Z-1/3)}+\dfrac{(1/2)Z}{(Z-1)}$

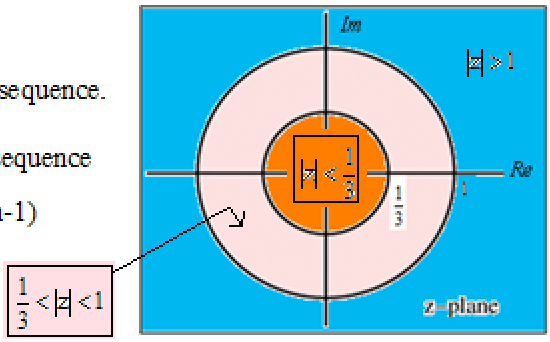
a) $\lvert z \rvert > 1$ ROC Since the ROC outside the poles The sequence is positive $x(n) = -(1/2)(1/3)^n\ u(n) + (1/2)\ u(n)$	b) $\lvert z \rvert < \frac{1}{3}$ ROC Since the ROC inside the poles The sequence is negative $x(n) = (1/2)(1/3)^n\ u(-n-1) - (1/2)\ u(-n-1)$

c) $\frac{1}{3} < \lvert z \rvert < 1$ ROC is

$\frac{-1/2Z}{(Z-1/3)} \longrightarrow$ outside pole, +ve sequence.

$\frac{1/2Z}{(Z-1)} \longrightarrow$ inside pole, -ve sequence

$x(n) = -(1/2)(1/3)^n\ u(n) - (1/2)\ u(-n-1)$



7-Properties of ROC of the Z-Transform

- ROC is a ring or disk centered at the origin.
- Fourier transform converges absolutely if ROC includes the unit circle.
- ROC contains no poles but is bounded by poles.
- If the sequence is finite in length, ROC is the Entire z-plane except possibly $z = 0$ and $z = \infty$.
- If the sequence is right-sided, ROC is an outer disk.
- If the sequence is left-sided, ROC is an inner disk.
- If the sequence is double-sided, ROC is a ring.
- ROC is a connected region.

Example- 7: Find the inverse Z-Transform of

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}} = \frac{\left(1+z^{-1}\right)^{2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} = 2+\frac{-9}{1-\frac{1}{2}z^{-1}}+\frac{8}{1-z^{-1}}$$

<table>
<tr><td>If ROC is $|z|>1$, then
$x(n)=2\delta(n)-9\left(\frac{1}{2}\right)^{n}u(n)+8u(n)$</td><td>If ROC is $\frac{1}{2}<|z|<1$, then
$x(n)=2\delta(n)-9\left(\frac{1}{2}\right)^{n}u(n)-8u(-n-1)$</td></tr>
<tr><td colspan="2">If ROC is $|z|<\frac{1}{2}$, then
$x(n)=2\delta(n)+9(1/2)^{n}u(-n-1)-8u(-n-1)$</td></tr>
</table>

Example- 8: find the inverse Z-Transform of

$$X(z)=\frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \qquad \text{ROC}:|z|>\frac{1}{2}$$

- Order of numerator is smaller than denominator (in terms of z^{-1})

- No higher order poles

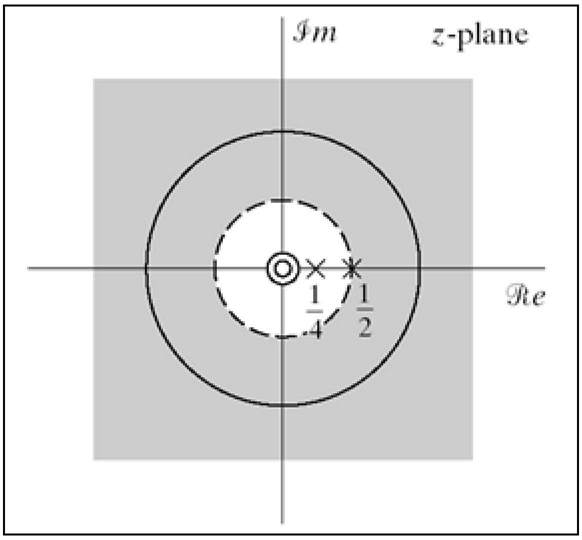
$$X(z)=\frac{A_1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)}$$

$$A_1=\left(1-\frac{1}{4}z^{-1}\right)X(z)\Big|_{z=\frac{1}{4}}=\frac{1}{\left(1-\frac{1}{2}\left(\frac{1}{4}\right)^{-1}\right)}=-1 \quad \text{and} \quad A_2=\left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}}=\frac{1}{\left(1-\frac{1}{4}\left(\frac{1}{2}\right)^{-1}\right)}=2$$

$$X(z)=\frac{-1}{\left(1-\frac{1}{4}z^{-1}\right)}+\frac{2}{\left(1-\frac{1}{2}z^{-1}\right)} \qquad |z|>\frac{1}{2}$$

- **ROC extends to infinity, - Indicates right sided sequences**

$$x(n)=2\left(\frac{1}{2}\right)^{n}u(n)-\left(\frac{1}{4}\right)^{n}u(n)$$



Example- 9: Find the inverse Z-Transforn

$$X(z)=\frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}=\frac{\left(1+z^{-1}\right)^{2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \qquad |z|>1$$

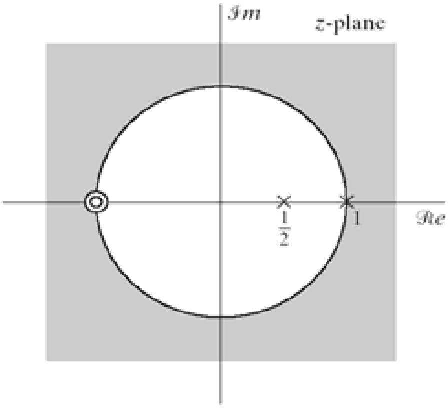
- Long division to obtain B$_o$

$$\begin{array}{r} 2 \\ \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 \overline{\big) z^{-2}+2z^{-1}+1} \\ \underline{z^{-2}-3z^{-1}+2} \\ 5z^{-1}-1 \end{array}$$

$$X(z)=2+\frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)} \quad \text{or} \quad X(z)=2+\frac{A_1}{1-\frac{1}{2}z^{-1}}+\frac{A_2}{1-z^{-1}}$$

$$A_1=\left(1-\frac{1}{2}z^{-1}\right)X(z)\Big|_{z=\frac{1}{2}}=-9 \qquad A_2=\left(1-z^{-1}\right)X(z)\Big|_{z=1}=8$$

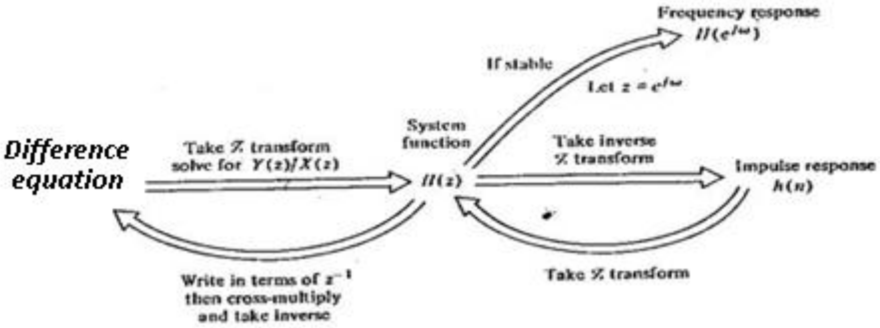
$$X(z)=2-\frac{9}{1-\frac{1}{2}z^{-1}}+\frac{8}{1-z^{-1}} \qquad |z|>1$$



- ROC extends to infinity, Indicates right-sides sequence

$$x(n)=2\delta(n)-9(1/2)^n\,u(n)\;-8\,u(n)$$

8- Relationships between System Representations:



Relationships between difference equation, system function, impulse response, and frequency response for stable causal systems represented by linear, constant coefficient difference equation.

Example- 10: Using Z-Transform, find the solution (*for* $n \geq 0$) to the following linear constant coefficient difference equation:

$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=\left(\frac{1}{4}\right)^{n}$

With initial conditions $y(-1)=4$ and $y(-2)=10$.

Solution: Taking the Z-transform of both sides gives

$$Y(Z)-\frac{3}{2}\{y(-1)+Z^{-1}\,Y(Z)\}+\frac{1}{2}\{y(-2)+Z^{-1}y(-1)+Z^{-2}Y(Z)\}=\frac{z}{z-\frac{1}{4}}$$

Substituting in the initial conditions and rearranging gives

$$Y(Z)=\left[1-\frac{3}{2}Z^{-1}+\frac{1}{2}Z^{-2}\right]=\frac{z}{z-\frac{1}{4}}+1-2Z^{-1}$$

And dividing by $\left[1-\frac{3}{2}Z^{-1}+\frac{1}{2}Z^{-2}\right]$

$$Y(Z)=\frac{2\left(2Z^{2}-\frac{9}{2}Z+\frac{1}{2}\right)}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(z-1)}$$

By partial fraction expansion, we can write

$$Y(Z)=\quad\frac{\frac{1}{3}z}{Z-\frac{1}{4}}+\frac{z}{Z-\frac{1}{2}}+\frac{\frac{2}{3}z}{z-1}$$

Taking the inverse z-transform, the difference equation is

$$y(n)=\left[\frac{1}{3}\left(\frac{1}{4}\right)^{n}+\left(\frac{1}{2}\right)^{n}+\frac{2}{3}\right]u(n)$$

H.W: Try to solve it in time domain and compare the results.

Example- 11: Given that $H(Z)=\frac{(Z+1)}{(Z^{2}-2Z+3)}$ Represents a causal system, find a difference equation realization and the frequency response of the system.

Solution: Since the system is causal, first write *H*(z) in terms of negative power of z

$$H(Z)=\frac{Y(z)}{X(z)}=\frac{(Z+1)}{(Z^{2}-2Z+3)}=\frac{Z^{-1}+Z^{-2}}{1-2Z^{-1}+3Z^{-2}}$$

Now Cross Multiply:

$$Y(z)(1-2Z^{-1}+3Z^{-2})=X(z)(Z^{-1}+Z^{-2})$$

Taking the inverse transform yields the following difference equation:

$$y(n)-2y(n-1)+3y(n-2)=x(n-1)+x(n-2)$$

The frequency response can be obtained by letting $Z=e^{j\omega}$ becomes

$$H\left(e^{j\omega}\right)=\left.\frac{(Z+1)}{(Z^2-2Z+3)}\right|\xrightarrow{Z=e^{j\omega}}=\frac{e^{j\omega}+1}{e^{2j\omega}-2e^{j\omega}+3}$$

$$e^{j\omega}=\cos\omega+j\sin\omega$$

$$H\left(e^{j\omega}\right)=\frac{(1+\cos\omega)+j\sin\omega}{\cos 2\omega-2\cos\omega+3+j(\sin 2\omega-\sin\omega)}$$

$$H\left(e^{j\omega}\right)=\left|H\left(e^{j\omega}\right)\right|\,.\,\underline{|\emptyset\left(e^{j\omega}\right)}$$

$$\left|H\left(e^{j\omega}\right)\right|=\frac{\sqrt{(1+\cos\omega)^2+(\sin\omega)^2}}{\sqrt{(3+\cos 2\omega-2\cos\omega)^2+(\sin 2\omega+\sin\omega)^2}}$$

$$\emptyset\left(e^{j\omega}\right)=\tan^{-1}\frac{\sin\omega}{1+\cos\omega}-\tan^{-1}\frac{\sin 2\omega+2\sin\omega}{3+\cos 2\omega-2\cos\omega}$$

DIGITAL FILTER DESIGN

Methods Of System Representation:

1-Difference equation realization:

$$\sum_{k=0}^{N} \hat{a}_k y(n-k) = \sum_{k=0}^{M} \hat{b}_k x(n-k)$$

General form D.E.

2-Transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

If a_o=1 & other a_k's=0, then the filter is FIR, otherwise it is IIR.

3-The impulse response:

$$h(n) = \mathcal{Z}^{-1}\{ H(z)\}$$

Digital Filter Specifications

1-Frequency Response

a-attenuation in pass band and stop band

b-cutoff frequency and roll off frequencies.

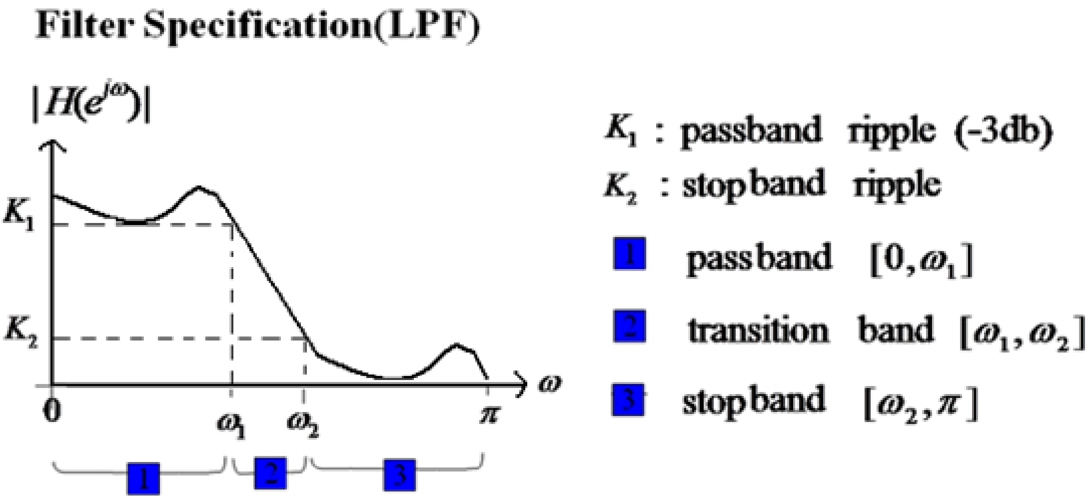
2- The magnitude and/or the phase (delay) response are specified for the design of a digital filter for most applications.

• In some situations, the unit sample response or the step response may be specified.

• In most practical applications, the problem of interest is the development of a realizable approximation to a given magnitude response specification

3- Phase response can be corrected by cascading the filter with an all-pass section!!!

For example, the magnitude response $\left|H\left(e^{j\omega}\right)\right|$ of a digital lowpass filter may be given as indicated below



- **As a result, filter specifications are given only for the frequency range 0≤ ω ≤ π**

To SIMULATE an analog filter, a discrete-time filter H(z) is used in the analog – to digital – H(z) – digital – to – analog structure shown in Fig. 2.

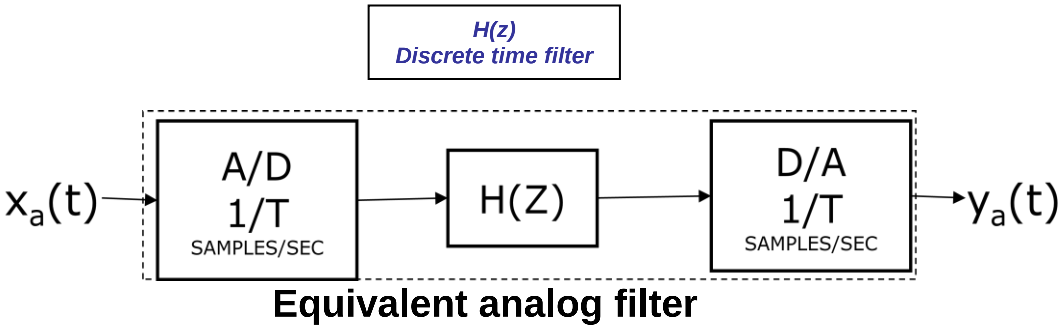


Fig. 2 Simulation of an analog filter

IIR FILTER DESIGN:

Several different techniques for designing H(z).

1- Numerical Methods.

2- Biliner Transformation Method .

3- Impulse-Invariant Method.

1- THE DESIGN BY USING NUMERICAL SOLUTIONS OF DIFFERNTIAL EQUATIONS:

Simulates a continuous-time linear filter specified by the following differential equation:

$$\sum_{k=0}^{N} C_k \frac{d^k y_{a\,(t)}}{dt^k} = \sum_{k=0}^{M} d_k \frac{d^k x_{a\,(t)}}{dt^k}$$

This filter has input $x_a(t)$ and output $y_a(t)$ and can be characterized by its system function $H_a(s)$ by taking the Laplace transform

$$H_a\ (s) = \frac{\sum_{k=0}^{M} d_k S^k}{\sum_{k=0}^{N} C_K S^k}$$

Suppose that we approximate the derivatives by backward differences. The first backward difference $\nabla^{(1)}[.]$ is defined by

$$\nabla^{(1)}[y(n)] = \frac{[y(n) - y(n-1)]}{T}$$

Higher-order backward difference are found by

$$\nabla^{(k)}[y(n)] = \nabla^{(1)}[\nabla^{(k-1)}\{y(n)\}]$$

Using the kth-order differences as approximations to the derivatives in Eq.(4-4) we have

$$\sum_{k=0}^{N} C_k \nabla^{(k)}[y_a(nT)] = \sum_{k=0}^{M} d_k\ \nabla^{(k)}[x_a(nT)]$$

The above equation represents a numerical approach for obtaining $y_a(nT)$, the sampled version of $y_a(t)$. The Z-transform of the first and kth-order difference are given below:

$$\overset{\top}{Z}\ \{\nabla^{(1)}[y(n)]\} = \overset{\top}{Z}\left\{\frac{[y(n)-y(n-1)]}{T}\right\} = Y(z)(1 - z^{-1})/T$$

$$\{\nabla^{(k)}[y(n)]\} = Y(z)\left(\frac{1 - z^{-1}}{T}\right)^k$$

The Z-transform of both sides

$$\sum_{k=0}^{N} C_k\,[\frac{1 - z^{-1}}{T}]^k\ Y(z) = \sum_{k=0}^{M} d_k\ [\frac{1 - z^{-1}}{T}]^k\ X(z)$$

The transfer function is easily seen to be:

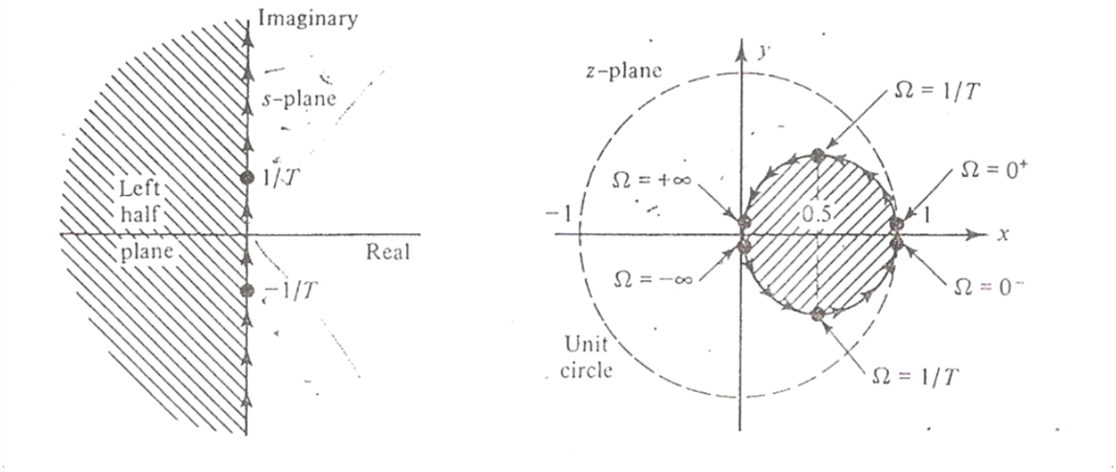
$$\boldsymbol{H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} d_k\ [\frac{1-z^{-1}}{T}]^k}{\sum_{k=0}^{N} C_k\ [\frac{1-z^{-1}}{T}]^k}}$$

Comparing with the Equation of $H_a(s)$, we see that H(z) can be obtaind by replacing S by $(1-z^{-1})/T$, that is

$$\boldsymbol{s=\frac{1-z^{-1}}{T}\ ,\quad \text{or}\quad z=\frac{1}{1-sT}}$$

As the frequency response for the analog system is obtained by letting s = j Ω , it is of interest to look at the image in the z-plane of the j Ω axis of the s-plane which is

$$\boldsymbol{z=\frac{1}{1-j\Omega T}=\frac{1}{1+\Omega^2T^2}+j\frac{\Omega T}{1+\Omega^2T^2}=x+jy}$$



In the above figure, the image of the $j\Omega$ axis of the s-plane in the z-plane for the mapping $z=\frac{1}{1-sT}$ is shown

It is easy to see that *x* (the real part of z) and *y* (the imaginary part of z) are related by

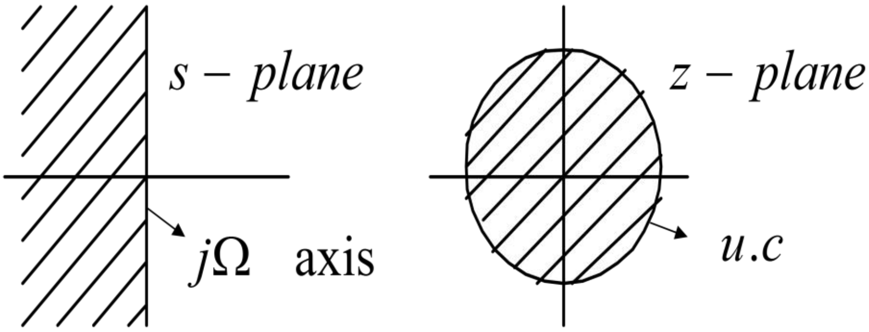
$$\boldsymbol{x^2+y^2=x}$$

Completing the square in the above equation gives the following equation:

$$\boldsymbol{(x-1/2)^2+y^2=1/4}$$

Thus, the image in the z-plane of the $j\Omega$ axis of the S-plane is a circle of radius 1/2 , as shown in the figure.

The frequency response of the digital filter is obtained by evaluating H(z) on the unit circle, z = $e^{j\omega}$. The shape of the equivalent frequency response of the H(z) would not be similar to that of $H_a(s)$. <u>To preserve the shape of the frequency response, we like to have the transformation from analog filter to digital filter take the jΩ axis of the s-plane into the unit circle in z-plane.</u>



Example -1: An analog filter with system function

$$H_a\ (s) = \frac{1}{(s+1)(s+2)}\ ,$$

a- **Find the H(z) using numerical method**

b- **Plot the frequency response for f_s = 5 bps?**

<u>Solution:</u>

$$H_a(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

By numerical method

a- **Since,** $s = \frac{1-z^{-1}}{T}$ **Then** $H(z) = \frac{1}{\left[\frac{1-z^{-1}}{T}\right]^2 + 3\left[\frac{1-z^{-1}}{T}\right] + 2}$

Or $$H(z) = \frac{T^2}{1-2z^{-1}+z^{-2}-3Tz^{-1}+2T^2}$$

b- f_s **= 5 bps**, $T^2 = 0.004$

$$H(e^{j\omega}) = H(z) \xrightarrow[z=e^{j\omega}]{} = \frac{T^2}{1-2e^{-j\omega}+e^{-2j\omega}-3Te^{-j\omega}+2T^2}$$

$$H(e^{j\omega}) = |H(e^{j\omega})|\ \underline{\lfloor H(e^{j\omega})}$$

***H.W.*:** **Suppose we are given the following differential equation:**

$$\sum_{k=0}^{N} C_k \frac{d^k y_{a\,(t)}}{dt^k} = \sum_{k=0}^{M} d_k \frac{d^k x_{a\,(t)}}{dt^k}$$

The first forward difference $\Delta^{(1)}[x(n)]$ is defined by

$$\Delta^{(1)}[x(n)] = \frac{[x(n+1) - x(n)]}{T}$$

and the n^th^ forward difference is obtained by successive first forward differences.

(a)- Find the mapping from the s-plane to the z-plane necessary to obtain the digital transfer function directly from the analog transfer function.

(b)- Using such transformation, find the z-plane image of s = $j\Omega$ as Ω goes from $-\infty$ to ∞ .

Solution:

(a)- The z-transform of the first forward difference is given by

$$Z\left[\Delta^{(1)}[x(n)]\right] = Z\left\{\frac{[x(n+1)-x(n)]}{T}\right\} = \left\{\frac{[zX(z)-X(z)]}{T}\right\} = \frac{(z-1)}{T}\,X(z)$$

The z-transform of the k^{th} - order forward difference is

$$Z\left[\Delta^{(k)}[x(n)]\right] = \left(\frac{z-1}{T}\right)^{k} X(z)$$

Using the k^{th} - order forward differences as approximations to the derivatives in the given differential equation, we have

$$\sum_{k=0}^{N} C_K\, \nabla^{(k)}[y_a(nT)] = \sum_{k=0}^{M} d_K\, \nabla^{(k)}[x_a(nT)]$$

The z-transform of both sides

$$\sum_{k=0}^{N} C_k\, \nabla^{(k)}[\frac{z-1}{T}]^{k}\, Y(z) = \sum_{k=0}^{M} d_k\, \nabla^{(k)}[\frac{z-1}{T}]^{k}\, X(z)$$

The transfer function is easily seen to be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} d_k\, \nabla^{(k)}[\frac{z-1}{T}]^k}{\sum_{k=0}^{N} C_k\, \nabla^{(k)}[\frac{z-1}{T}]^k}$$

Using Laplace transform on the given differential equation, we can write

$$H_a(s) = \frac{\sum_{k=0}^{M} d_K S^k}{\sum_{k=0}^{N} C_K S^k}$$

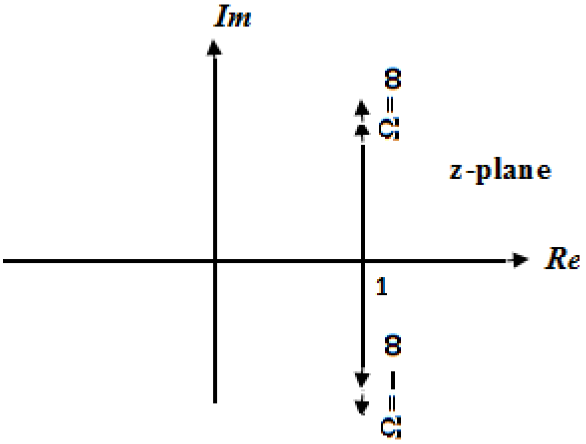
Comparing the above to equations we see that the mapping from the s-plane to z-plane is

$$s = \frac{z-1}{T}\ , \qquad \text{or} \quad z = 1 + sT$$

(b)- for $s = j\Omega$ *as* Ω goes from $-\infty$ *to* ∞ .

$s = \frac{z-1}{T}$, $\quad z = 1 + sT$ **and for $s = j\Omega$, we get** $\quad z = 1 + j\Omega T$

The locus of points in the z-plane as Ω varies is shown below



2- IIR FILTER DESIGN BY BILINEAR TRANSFORMATION

Design Concept:- Using a first-order differential equation

$$a_1\, y'_a(t) + a_0 y_a(t) = b_0 x(t) \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

The transfer function $H_a(s)$ can be written as

$$H_a(s) = \frac{b_0}{a_1\, s + a_0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

The fundamental theorem of integral calculus allows us to write

$$y_a(t) = \int_{t_0}^{t} y'_a(t)\, dt + y_a(t_0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Since Eq.(3) holds for any t with any t_0 , we let $t = nT$ and $t_0 = (n-1)T$ to get

$$y_a(nT) = \int_{(n-1)T}^{nT} y'_a(t)\, dt + y_a[(n-1)T] \quad \ldots\ldots\ldots\ldots(4)$$

Using the trapezoidal rule $\displaystyle\int_{x_1}^{x_2} f(x)\, dx \approx \frac{1}{2}(x_2 - x_1)\,[f(x_2) + f(x_1)]$

To approximate the integral and by assuming equality, a recursive relationship for determining $y_a(nT)$ can be found from Eq. (4) as follows:

$$y_a(nT) = y_a[(n-1)T] + \left(\frac{T}{2}\right)\{y'_a(nT) + y'_a[(n-1)T]\} \ldots..(5)$$

From Eq. (2)

$$y'_a(t) = -\frac{a_0}{a_1}y_a(t) + \frac{b_0}{a_1}x(t) \qquad \ldots\ldots\ldots\ldots\ldots(6)$$

Now the differential equation evaluated in *t* = *nT*:

$$y'_a(nT) = -\frac{a_0}{a_1}y_a(nT) + \frac{b_0}{a_1}x(nT) \ \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

&

$$y'_a\big((n-1)T\big) = -\frac{a_0}{a_1}y_a\big((n-1)T\big) + \frac{b_0}{a_1}x\big((n-1)T\big) \ldots..(8)$$

Substitute (7) & (8) into (5) to obtain a difference equation for the equivalent discrete-time system. Then

$$y_a(nT) = \left(\frac{T}{2}\right)\left\{-\frac{a_0}{a_1}y_a(nT) + \frac{b_0}{a_1}x(nT) - \frac{a_0}{a_1}y_a\big((n-1)T\big) + \frac{b_0}{a_1}x\big((n-1)T\big)\right\} + y_a[(n-1)T]$$

The previous expression can be expressed in the following form

$$\left(1+\frac{Ta_0}{2a_1}\right)y_a(nT) - \left(1-\frac{Ta_0}{2a_1}\right)y_a\big((n-1)T\big) = \frac{Tb_0}{2a_1}\big\{x(nT) + x\big((n-1)T\big)\big\}$$

The z-transform of the previous equation is

$$\left(1+\frac{Ta_0}{2a_1}\right)Y(z) - (1-\frac{Ta_0}{2a_1})z^{-1}Y(z) = \frac{Tb_0}{2a_1}\{X(z) + z^{-1}X(z)\}$$

Transfer function of the equivalent digital filter is

$$H(z) = \frac{Y(z)}{X(z)} \quad \text{or} \quad H(z) = \frac{\frac{Tb_0}{2a_1}(1+z^{-1})}{\left(1+\frac{Ta_0}{2a_1}\right)-\left(1-\frac{Ta_0}{2a_1}\right)z^{-1}}$$

$$H(z) = \frac{\frac{Tb_0}{2a_1}(1+z^{-1})}{\left(1+\frac{Ta_0}{2a_1}\right)-\left(1-\frac{Ta_0}{2a_1}\right)z^{-1}} \text{ or } H(z) = \frac{\frac{Tb_0}{2a_1}(1+z^{-1})}{(1-z^{-1}) + \frac{Ta_0}{2a_1}(1+z^{-1})}$$

$$H(z) = \frac{\frac{Tb_0}{2a_1}}{\frac{(1-z^{-1})}{(\,1+z^{-1})} + \frac{Ta_0}{2a_1}} \qquad \text{Thus } H(z) = \frac{b_0}{a_1\,\frac{2(1-z^{-1})}{T(\,1+z^{-1})} + a_0}$$

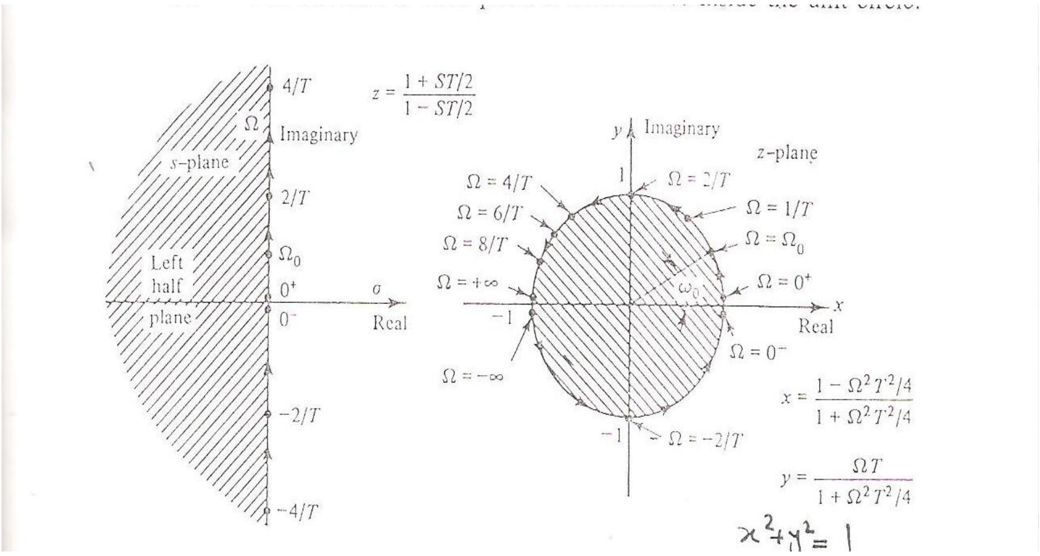
Comparing $H_a(s)$ of Eq. (2) and H(z) it can be seen that H(z) can be obtained from $H_a(s)$ by using the following mapping relation:

$$s = \frac{2}{T}\frac{(1-z^{-1})}{(\,1+z^{-1})} \quad \text{or} \quad z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}} \quad \text{.......(18)}$$

This is the bilinear transformation. The image of the jΩ axis from the s-plane in the z-plane is shown in the Figure below. The bilinear transformation given in Eq.(18) has the following properties:

(1)- The entire *j*Ω axis of the s-plane goes into the unit circle of the z-plane.

(2)- The left half side of the s-plane is transformed inside the uint circle of the z-plane.



The image in the z-plane of the jΩ axis of the s-plane for **the mapping**

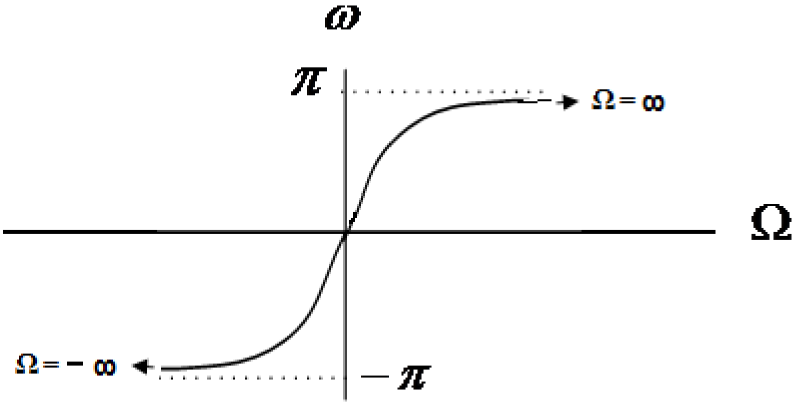
$$z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$$

Therefore a stable analog filter would be transformed into a stable digital filter. While the frequency responses of analog filter and digital filter have the same amplitudes. There is *<u>a nonlinear relationship</u>* between corresponding digital and analog frequencies.

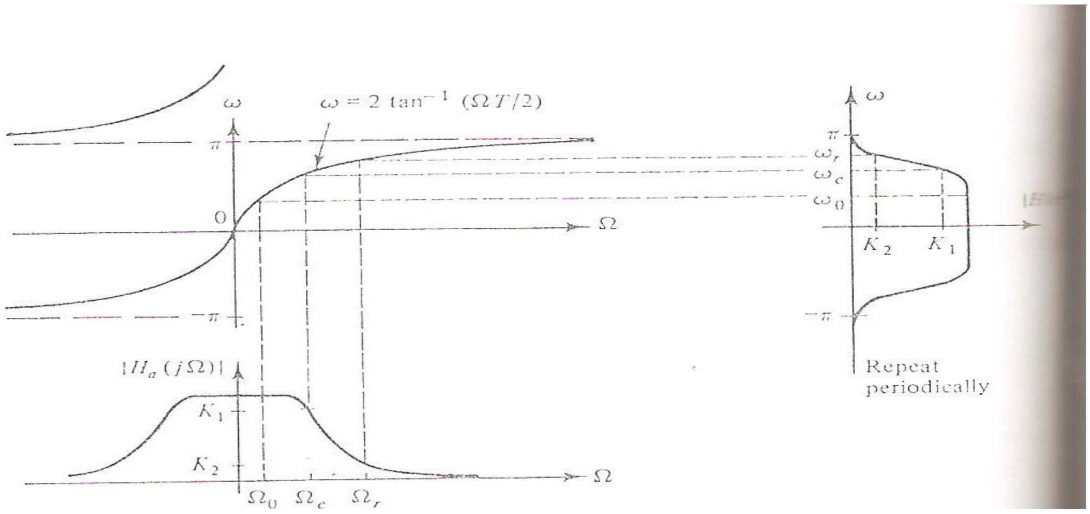
If $s = j\Omega$, then $z = \frac{1+\frac{j\Omega T}{2}}{1-\frac{j\Omega T}{2}}$

$$\boldsymbol{if z = e^{\omega}, s = \frac{2}{T}\frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{2}{T}\frac{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}}-e^{-\frac{j\omega}{2}}\right)}{e^{-\frac{j\omega}{2}}\left(e^{\frac{j\omega}{2}}+e^{-\frac{j\omega}{2}}\right)} = \frac{2}{T}\frac{j2\sin\frac{\omega}{2}}{2\cos\frac{\omega}{2}} = j\frac{2}{T}\tan\frac{\omega}{2} = j\Omega,}$$

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \qquad \text{or} \qquad \omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right)$$



- ***<u>Nonlinear mapping introduces a distortion in the frequency axis called frequency warping seethe following Figure:</u>***



The digital frequency will have a critical frequency $\boldsymbol{\omega_c}$ given by $\boldsymbol{\omega_c = 2\tan^{-1}(\Omega_c T/2)}$. The equivalent critical frequency becomes

$$\boldsymbol{\Omega_{c\,eq} = \frac{2}{T}\tan^{-1}(\Omega_c T/2)}$$

<u>WHICH WILL GIVE</u> Ω_c ***ONLY IF***
$(\boldsymbol{\Omega_c T/2})$ *<u>is small that</u>* $\boldsymbol{\tan^{-1}(\Omega_c T/2)}$ *<u>is approximately equal to</u>* $(\boldsymbol{\Omega_c T/2})$

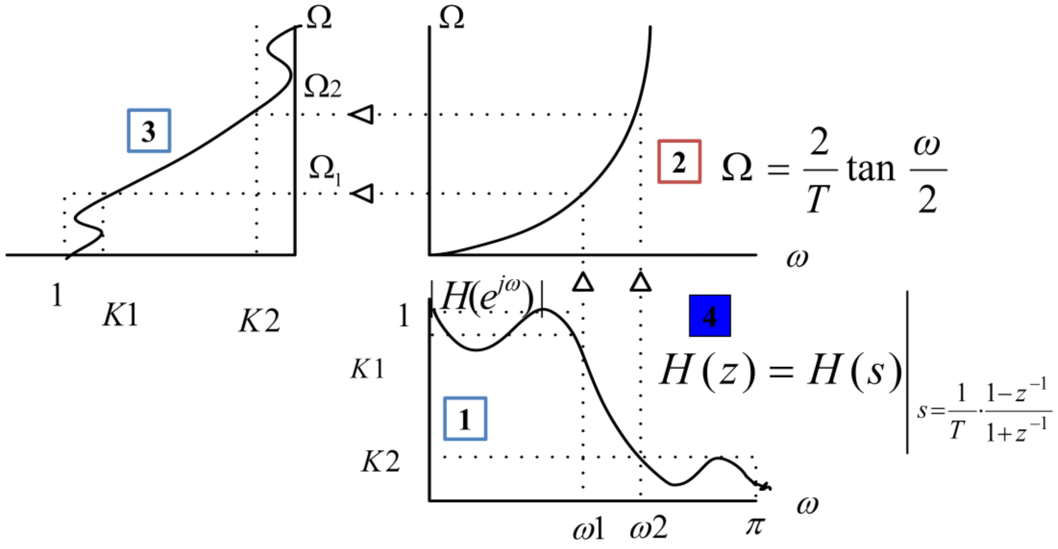
This warping of the critical frequency will be compensated for in the design procedure using the bilinear transformation by *<u>prewarping.</u>*

* IIR Filter Design Procedure

TABLE 3-2

TABLE 3.1

- **1** Given specification in digital domain
- **2** Convert it into analog filter specification
- **3** Design analog filter (Butterworth, Chebyshov, elliptic):H(s)
- **4** Apply bilinear transform to get H(z) out of H(s)



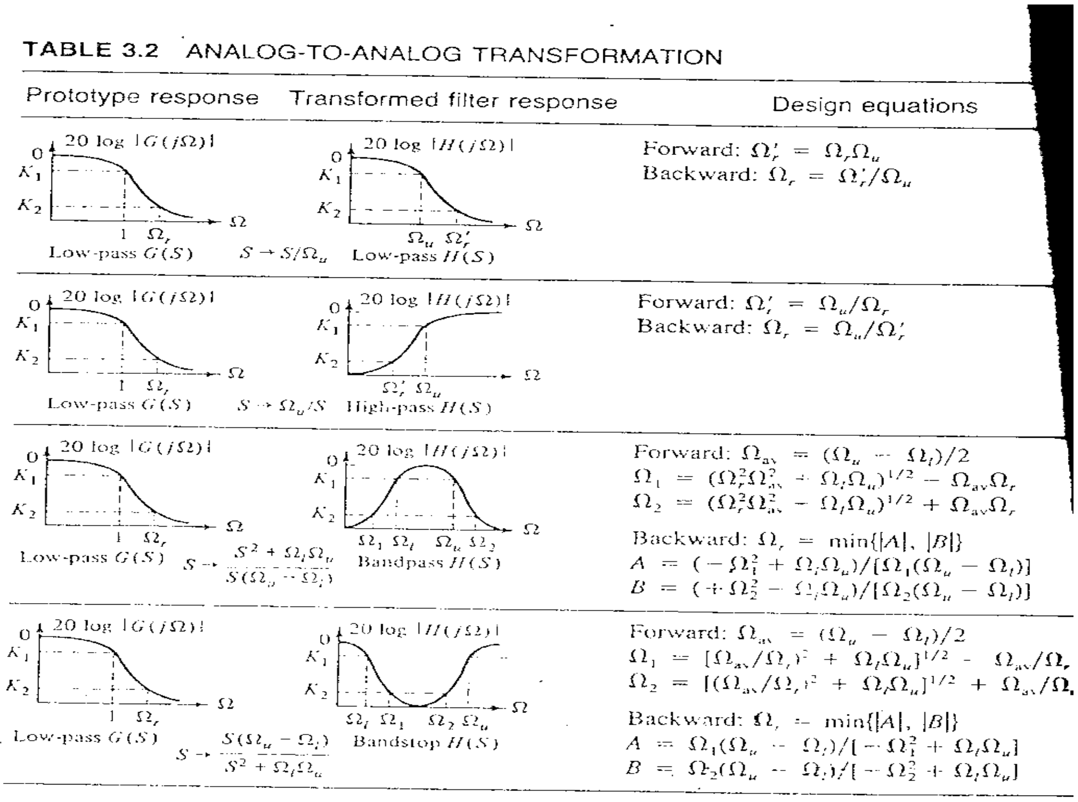


TABLE 3.2 ANALOG-TO-ANALOG TRANSFORMATION

Prototype response		Transformed filter response	Design equations
$20\log\|G(j\Omega)\|$ Low-pass $G(S)$	$S \rightarrow S/\Omega_u$	$20\log\|H(j\Omega)\|$ Low-pass $H(S)$	Forward: $\Omega_r' = \Omega_r\Omega_u$ Backward: $\Omega_r = \Omega_r'/\Omega_u$
$20\log\|G(j\Omega)\|$ Low-pass $G(S)$	$S \rightarrow \Omega_u/S$	$20\log\|H(j\Omega)\|$ High-pass $H(S)$	Forward: $\Omega_r' = \Omega_u/\Omega_r$ Backward: $\Omega_r = \Omega_u/\Omega_r'$
$20\log\|G(j\Omega)\|$ Low-pass $G(S)$	$S \rightarrow \frac{S^2 + \Omega_l\Omega_u}{S(\Omega_u - \Omega_l)}$	$20\log\|H(j\Omega)\|$ Bandpass $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} - \Omega_{av}\Omega_r$ $\Omega_2 = (\Omega_r^2\Omega_{av}^2 - \Omega_l\Omega_u)^{1/2} + \Omega_{av}\Omega_r$ Backward: $\Omega_r = \min\{\|A\|, \|B\|\}$ $A = (-\Omega_1^2 + \Omega_l\Omega_u)/[\Omega_1(\Omega_u - \Omega_l)]$ $B = (+\Omega_2^2 - \Omega_l\Omega_u)/[\Omega_2(\Omega_u - \Omega_l)]$
$20\log\|G(j\Omega)\|$ Low-pass $G(S)$	$S \rightarrow \frac{S(\Omega_u - \Omega_l)}{S^2 + \Omega_l\Omega_u}$	$20\log\|H(j\Omega)\|$ Bandstop $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = [\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} - \Omega_{av}/\Omega_r$ $\Omega_2 = [(\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} + \Omega_{av}/\Omega_r$ Backward: $\Omega_r = \min\{\|A\|, \|B\|\}$ $A = \Omega_1(\Omega_u - \Omega_l)/[-\Omega_1^2 + \Omega_l\Omega_u]$ $B = \Omega_2(\Omega_u - \Omega_l)/[-\Omega_2^2 + \Omega_l\Omega_u]$

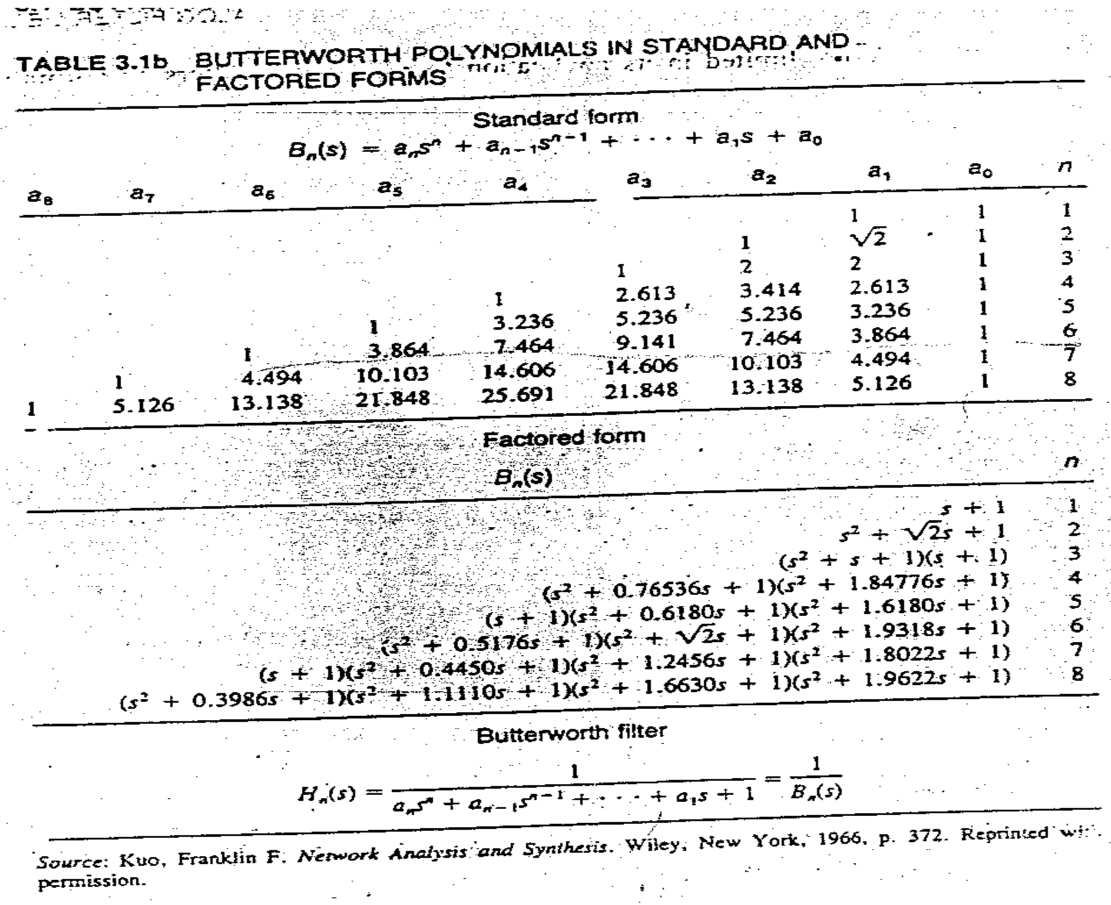


TABLE 3.1b BUTTERWORTH POLYNOMIALS IN STANDARD AND FACTORED FORMS

Standard form

$$B_n(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0	n
							1	1	1
						1	$\sqrt{2}$	1	2
					1	2	2	1	3
				1	2.613	3.414	2.613	1	4
			1	3.236	5.236	5.236	3.236	1	5
		1	3.864	7.464	9.141	7.464	3.864	1	6
	1	4.494	10.103	14.606	14.606	10.103	4.494	1	7
1	5.126	13.138	21.848	25.691	21.848	13.138	5.126	1	8

Factored form

$B_n(s)$	n
$s + 1$	1
$s^2 + \sqrt{2}s + 1$	2
$(s^2 + s + 1)(s + 1)$	3
$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$	4
$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$	5
$(s^2 + 0.5176s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 1.9318s + 1)$	6
$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2456s + 1)(s^2 + 1.8022s + 1)$	7
$(s^2 + 0.3986s + 1)(s^2 + 1.1110s + 1)(s^2 + 1.6630s + 1)(s^2 + 1.9622s + 1)$	8

Butterworth filter

$$H_n(s) = \frac{1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1} = \frac{1}{B_n(s)}$$

Source: Kuo, Franklin F. *Network Analysis and Synthesis*. Wiley, New York, 1966, p. 372. Reprinted with permission.

Example (LP Butterworth Filter)

Design and realize a digital low-pass filter using the bilinear transformation method to satisfy the following characteristics:

(a) Monotonic stopband and passband(*Butterworth Filter*).
(b) -3.01dB cutoff frequency of 0.5π rad.
(c) Magnitude down at least 15 dB at 0.75 π rad.

Solution:-

The design procedure is that of using the bilinear transformation on an analog prototype and consists of the following steps:

Step1. **Prewarp the critical digital frequencies**
$\omega 1 = 0.5\,\pi$ *and* $\omega 2 = 0.75\,\pi$ **using T=1 sec to get**

$$\Omega_1' = \frac{2}{T}\tan\frac{\omega 1}{2} = 2\tan\frac{0.5\,\pi}{2} = 2.00$$

$$\Omega_2' = \frac{2}{T}\tan\frac{\omega 2}{2} = 2\tan\frac{0.75\,\pi}{2} = \mathbf{4.8282}$$

Step2. **Design an analog low-pass filter with critical frequencies Ω_1' and Ω_2' *that satisfy***

$$2 \geq 20log\,|H_a(j\Omega_1')| \geq\; -3.01\,dB = K_1$$

$$20log\,|H_a(j\Omega_2')| \geq\; -15\,dB = K_2$$

<u>A Butterwirth filter</u> **is used to satisfy the monotonic property and has an order n and critical frequency Ω_c** ***<u>Determined by the following equations :</u>***

$$n = \left\lceil \frac{log_{10}\left[\left(10^{-k_1/10}-1\right)/\left(10^{-k_2/10}-1\right)\right]}{2log_{10}(\Omega_1/\Omega_2)} \right\rceil$$

$$\Omega_c = \Omega_1/\left(10^{-k_1/10}-1\right)^{1/2n}$$

We get

$$n = \left\lceil \frac{log_{10}\left[\left(10^{3.01/10}-1\right)/\left(10^{15/10}-1\right)\right]}{2log_{10}\left(\frac{2}{4.8282}\right)} \right\rceil = 1,9412 = 2$$

$$\Omega_c = 2/\left(10^{3,01/10}-1\right)^{1/4} = 2$$

Therefore the required prewarped analog filter using the Butterworth Table 3.1 and low-pass to low pass transformation from table 3.2 is

$$H_a(s) = \left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s\to s/2} = \frac{1}{(s/2)^2+\sqrt{2}(s/2)+1}$$

$$\frac{4}{s^2+2\sqrt{2}s+4}$$

Step3. **Applying the bilinear transformation method(T=1) to satisfy the given digital requirements:**

$$H(z) = H_a(s)|_{s\to\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]}$$

$$\boldsymbol{H(z) = \frac{4}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]^2 + 2\sqrt{2}\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right] + 4}}$$

$$\boldsymbol{H(z) = \frac{1+2z^{-1}+z^{-2}}{3.41+0.586z^{-2}}}$$

This digital filter can be realized by specification of a difference equation obtained from the transfer function H(z) given by

$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = \frac{1+2z^{-1}+z^{-2}}{3.41+0.586z^{-2}}}$$

Cross multiplying gives:

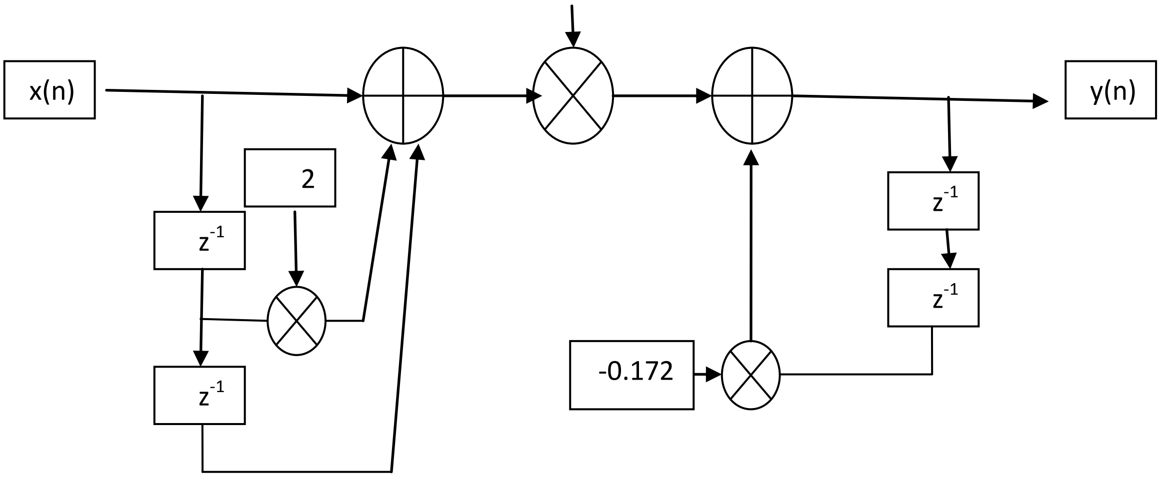
$$\boldsymbol{Y(z)[3.41+0.586z^{-2}] = X(z)[1+2z^{-1}+z^{-2}]}$$

And taking the inverse Z-transform we find

$$\boldsymbol{[3.41y(n)+0.586y(n-2)] = [x(n)+2x(n-1)+x(n-2)]}$$

By rearranging and scaling. y(n) can be realized by the following difference equation:

$$\boldsymbol{y(n) = 0.293[x(n)+2x(n-1)+x(n-2)] - 0.172y(n-2)}$$



Example (BP Butterworth Filter)

Design a digital band-pass filter representing an analog one with the following specifications:

(a) Stop band attenuation of at least 20dB at warpped frequencies of 20Hz and 45kHz.
(b) -3 dB lower and upper cutoff bilinear warpped frequencies of 50Hz and 20kHz.
(c) a monotonic frequencies.

Solution:-
The desired frequency response is shown on the right

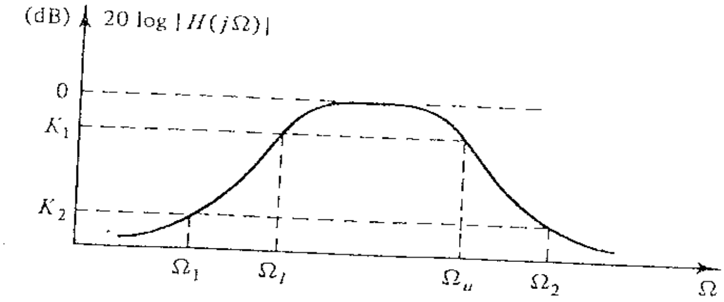


Figure 3.8 Typical bandpass requirements.

From table (3.2) we see

$$\Omega_1 = 2\pi(20) = 125.66\frac{rad}{sec}$$
$$\Omega_l = 2\pi(50) = 314.159\frac{rad}{sec}$$
$$\Omega_u = 2\pi(20000) = 125663\frac{rad}{sec}$$
$$\Omega_2 = 2\pi(45000) = 282743\frac{rad}{sec}$$

$$k_1 = -3dB \qquad k_2 = -20dB,$$

The Backward equation then gives the Ω_r for a normalized Low pass prototype.

$$\Omega_r = min[(|A|),(|B|)\ ]$$

From table (3.2) USING ($\Omega_1, \quad \Omega_2\ , \ \Omega_l\ and\ \Omega_u$)we see

$$A = \frac{(-\Omega_1^2 + \ \Omega_l\ \Omega_u)}{[\ \ \Omega_1(\ \Omega_u - \ \Omega_l]} \qquad B = \frac{(-\Omega_2^2 - \ \Omega_l\ \Omega_u)}{[\ \ \Omega_2(\ \Omega_u - \ \Omega_l]}$$

A=2.5053,   B=2.2545

The most critical value Ω_r ***is the minimum of the two, that is,*** $\Omega_r = min[(|A|),(|B|)\ ]$**=2.2545**

<u>The low-pass Butterworth filter</u>* of order** n **can then be easily calculated from *<u>the following equations :</u>

$$n = \left\lceil \frac{log_{10}\left[\left(10^{-k_1/10} - 1\right)/\left(10^{-k_2/10} - 1\right)\right]}{2log_{10}(1/\Omega_r)} \right\rceil$$

We get

$$n = \left\lceil \frac{log_{10}\left[\left(10^{3./10} - 1\right)/\left(10^{20/10} - 1\right)\right]}{2log_{10}\left(\frac{1}{2.2545}\right)} \right\rceil = \lceil 2.829 \rceil = 3$$

from the Butterworth Table 3.1b and n=3 we have the low-pass prototype as

$$H_{LP} = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

The required analog-to-analog transformation (table 3.2) is determined from $\Omega_l\ and\ \Omega_u$ ***as***

$$s \to \left[\frac{(s^2 + \ \ \Omega_u\,\Omega_l)}{s(\ \ \Omega_u - \ \Omega_l)}\right] = \frac{s^2 + 3.95 \times 10^7}{s(1.25 \times 10^5)}$$

$H_{BP}(s)$ ***then is finlly seen to be***

$$H_{BP}(s) = \frac{1}{\left[\frac{s^2 + 3.95 \times 10^7}{s(1.25 \times 10^5)}\right]^3 + 2\left[\frac{s^2 + 3.95 \times 10^7}{s(1.25 \times 10^5)}\right]^2 + 2\left[\frac{s^2 + 3.95 \times 10^7}{s(1.25 \times 10^5)}\right]^1 + 1}$$

$$H_{BP}(s) = \frac{1.9 \times 10^{15}s^3}{s^6 + 2.5 \times 10^5 s^5 + 3.15 \times 10^{10} s^4 + 1.99 \times 10^{15} s^3 + 1.25 \times 10^{18} s^2 + 3.9 \times 10^{20} s + 6.15 \times 10^{22}}$$

Applying the bilinear transformation method(T=1) to satisfy the given digital requirements:

$$H(z) = H_{BP}(s)\Big|_{s\to\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]}$$

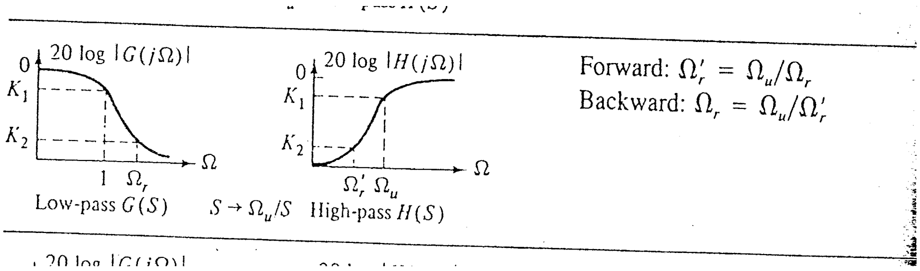
$$H(z) = \frac{1.9\times 10^{15}\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]^{3}}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]^{6} + 2.5\times 10^{5}\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]^{5} + 3.15\times 10^{10}\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]^{4} + 1.99\times 10^{15}\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]^{3} + 1.25\times 10^{18}\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]^{2} + 3.9\times 10^{20}\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right] + 6.15\times 10^{22}}$$

Example (HP Butterworth Filter)

Design a digital filter representing an analog one with the following specifications:

- **(a) Pass all signals of bilinear warpped frequencies greater than 200rad/sec with no more than 2dB of attenuation.**
- **(b) Stop band attenuation of greater than -20 dB at warpped d frequencies less than 100rad/sec.**
- **(c) a maximally flat IIR response.**

Solution:-
The desired frequency response is shown on the right



From table (3.2) we see

$$\Omega_u = 200\frac{rad}{sec},\quad \Omega_r' = 100\frac{rad}{sec}$$

$$k_1 = -2dB \qquad k_2 = -20dB,$$

The Backward equation then gives the Ω_r for a normalized Low pass prototype.

$\Omega_r = \ \Omega_u/\Omega_r'$**=200/100=2**

The low-pass Butterworth filter **now has the following specifications**

$$\Omega_1 = 1 \ , \qquad k_1 = -2dB$$

$$\Omega_2 = \ \Omega_r = 2 \ , \qquad k_2 = -20dB$$

The order of the filter is determined as follows

$$n = \left\lceil \frac{log_{10}\left[\left(10^{-k_1/10} - 1\right)/\left(10^{-k_2/10} - 1\right)\right]}{2log_{10}(\Omega_1/\Omega_2)} \right\rceil$$

We get

$$n = \left\lceil \frac{log_{10}\left[\left(10^{4/10} - 1\right)/\left(10^{20/10} - 1\right)\right]}{2log_{10}\left(\frac{1}{2}\right)} \right\rceil = \lceil 3.7 \rceil = 4$$

$$\Omega_c = \frac{\Omega_1}{\left(10^{-\frac{k_1}{10}} - 1\right)^{\frac{1}{2n}}} = \frac{1}{\left(10^{\frac{2}{10}} - 1\right)^{\frac{1}{8}}} = 1.069$$

from the Butterworth Table 3.1b and n=4 we have the *<u>NORMALIZED low-pass filter</u>* as

$$H_4 = \frac{1}{(s^2 + 0.765s + 1)\,(s^2 + 1.85s + 1)}$$

The low-pass filter prototype

$$H_{LP}(s) = H_4(s)\big]_{s \to \left[\frac{s}{\Omega_c}\right]}$$

$$H_{LP}(s) = \frac{1}{\left(\left(\frac{s}{1.069}\right)^2 + 0.765\left(\frac{s}{1.069}\right) + 1\right)\left(\left(\frac{s}{1.069}\right)^2 + 1.85\left(\frac{s}{1.069}\right) + 1\right)}$$

To get desired HPF apply LP TO HP(analog-to-analog transformation (table 3.2))

$$\boldsymbol{H_{BP}(s) = H_{LP}(s)]_{s\to\left[\frac{\Omega_u}{s}\right]=200/s}}$$

$\boldsymbol{H_{HP}(s)}$ ***then is finlly seen to be***

$$\boldsymbol{H_{BP}(s) = \frac{1}{\left((\frac{187}{s})^2 + 0.765(\frac{187}{s}) + 1\right)\left((\frac{187}{s})^2 + 1.85(\frac{187}{s}) + 1\right)}}$$

Applying the bilinear transformation method(T=1) to satisfy the given digital requirements:

$$\boldsymbol{H(z) = H_{BP}(s)|_{s\to\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]}}$$

$$\boldsymbol{H(z) = \frac{1}{\left((\frac{187}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]})^2 + 0.765(\frac{187}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]}) + 1\right)\left((\frac{187}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]})^2 + 1.85(\frac{187}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]}) + 1\right)}}$$

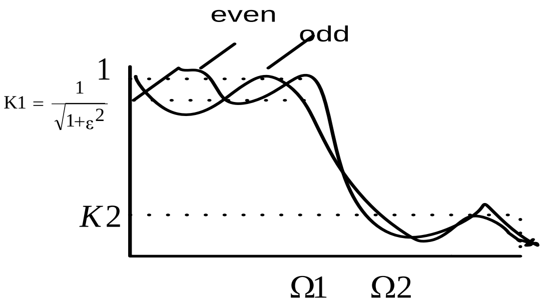
Example (Chebyshev LPF)

Design a *Chebyshev LPF* representing an analog one with the following specifications:

- **(a) Acceptable pass-band ripples of 2dB.**
- **(b) Cut off frequency of 40 rad/sec.**
- **(c) Stop band attenuation of 20 dB or more at 52 rad/sec.**

Solution:-

The desired frequency response is shown on the right



The general approach is to first change the requirements to those of a low-pass unit bandwidth PROTOTYPE, design such a LPF , and then apply a LP to LP Transformation to that PROTOTYPE.

From table (3.2) we see

$$\Omega_u = 52\frac{rad}{sec},\quad \Omega'_r = 40\frac{rad}{sec}$$

$$k_1 = 20log\,\frac{1}{\sqrt{1+\in^2}} = -2dB \qquad k_2 = -20dB,$$

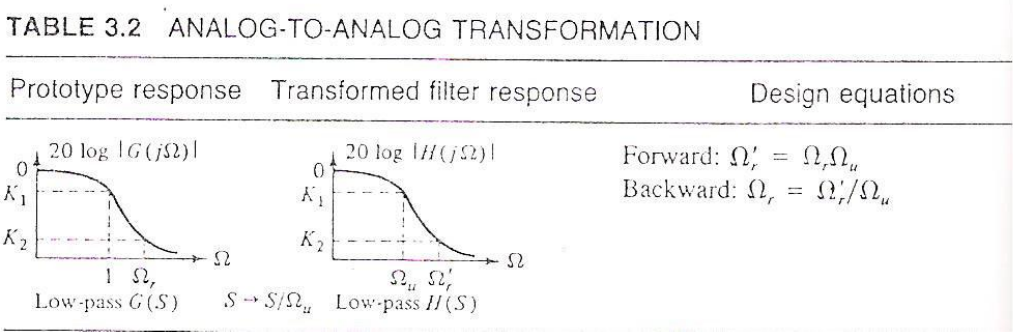


TABLE 3.2 ANALOG-TO-ANALOG TRANSFORMATION

Prototype response	Transformed filter response	Design equations
Low-pass $G(S)$	$S \to S/\Omega_u$ Low-pass $H(S)$	Forward: $\Omega'_r = \Omega_r\Omega_u$ Backward: $\Omega_r = \Omega'_r/\Omega_u$

The Backward equation then gives the Ω_r for a normalized Low pass prototype.

$\Omega_r = \quad \Omega_u/\Omega_r'$**=52/40=1.3 rad. /sec**

***The low-pass filter* now has the following specifications**

$$\boldsymbol{\Omega_1 = 1 \quad , \qquad k_1 = 20log\,\frac{1}{\sqrt{1+\epsilon^2}} = -2dB}$$

$$\boldsymbol{\Omega_2 = \quad \Omega_r = 1.3 \quad , \qquad k_2 = 20log\left(\frac{1}{A}\right) = -20dB}$$

<u>The low-pass Chebyshev filter</u>* of order n can then be easily calculated from *<u>the following equations :</u>

$$\boldsymbol{n = \left\lceil\frac{log_{10}\left[\left(g+\sqrt{g^2-1}\right)\right]}{log_{10}\left(\Omega_r+\sqrt{{\Omega_r}^2+1}\right)}\right\rceil} \text{ where } \boldsymbol{g = \sqrt{\frac{A^2-1}{\epsilon^2}},\ A = \frac{1}{|H(j\Omega|}}$$

$$\boldsymbol{k_1 = 20log\,\frac{1}{\sqrt{1+\epsilon^2}} = -2dB, \qquad \epsilon^2 = 0.58489}$$

$$\boldsymbol{k_2 = 20log\left(\frac{1}{A}\right) = -20dB, \qquad A = 10}$$

$$\boldsymbol{g = \sqrt{\frac{A^2-1}{\epsilon^2}} = 13}$$

$$\boldsymbol{n = \left\lceil\frac{log_{10}\left[\left(g+\sqrt{g^2-1}\right)\right]}{log_{10}\left(\Omega_r+\sqrt{{\Omega_r}^2+1}\right)}\right\rceil = \left\lceil\frac{log_{10}\left[\left(13+\sqrt{13^2-1}\right)\right]}{log_{10}\left(1.3+\sqrt{1.3^2+1}\right)}\right\rceil = 5}$$

Using the 2dB ripple part of Table 3.4

$$H_n(s) = \frac{k_n}{V_n(s)}, \qquad k_n = \begin{cases} \dfrac{b_0}{(1+\epsilon^2)^{\frac{1}{2}}} for \quad n \quad even \\ b_0 \; for \; n \; odd \end{cases}$$

$$V_n(s) = s^n + b_{n-1}s^{n-1} + \cdots + b_1 s^1 + b_0$$

for n=5, and the fact that <u>since n is odd</u>, we have the desired Chebyshev unit bandwidth low-pass filter as

$$H_5(s) = \frac{b_0}{s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s^1 + b_0}$$

$$H_{LP}(s) = H_5(s)\rfloor_{s \to s/40}$$

$$= \frac{b_0}{(s/40)^5 + b_4(s/40)^4 + b_3(s/40)^3 + b_2(s/40)^2 + b_1(s/40)^1 + b_0}$$

$$H_{LP}(z) = H_{lp}(s)\rfloor_{s \to 2(\frac{1-z}{1+z})}$$

(H.W)

Design a digital filter having a 1-dB cut off frequency at 75 Hz and greater than 20-dB attenuation for $> 50\ Hz$. Find H(z) that will satisfy the above prewarpped specifications for :

(a) Butterworth .

(b) Chebyshev approximations.

Chebyshev filter $H_n(s) = \frac{K_n}{V_n(s)}$ $K_n = \begin{cases} b_0/(1 + \epsilon^2)^{1/2} & \text{for } n \text{ even} \\ b_0 & \text{for } n \text{ odd} \end{cases}$

$V_n(s) = s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0$

a. ½ dB Ripple(ϵ = 0.3493114, ϵ^2 = 0.1220184)

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
1	2.8627752									
2	1.5162026	1.4256245								
3	0.7156938	1.5348954	1.2529130							
4	0.3790506	1.0254553	1.7168662	1.1973856						
5	0.1789234	0.7525181	1.3095747	1.9373675	1.1724909					
6	0.0947626	0.4323669	1.1718613	1.5897635	2.1718446	1.1591761				
7	0.0447309	0.2820722	0.7556511	1.6479029	1.8694079	2.4126510	1.1512176			
8	0.0236907	0.1525444	0.5735604	1.1485894	2.1840154	2.1492173	2.6567498	1.1460801		
9	0.0111827	0.0941198	0.3408193	0.9836199	1.6113880	2.7814990	2.4293297	2.9027337	1.1425705	
10	0.0059227	0.0492855	0.2372688	0.6269689	1.5274307	2.1442372	3.4409268	2.7097415	3.1498757	1.1400

b. 1-dB Ripple (ϵ = 0.5088471, ϵ^2 = 0.2589254)

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
1	1.9652267									
2	1.1025103	1.0977343								
3	0.4913067	1.2384092	0.9883412							
4	0.2756276	0.7426194	1.4539248	0.9528114						
5	0.1228267	0.5805342	0.9743961	1.6888160	0.9368201					
6	0.0689069	0.3070808	0.9393461	1.2021409	1.9308256	0.9282510				
7	0.0307066	0.2136712	0.5486192	1.3575440	1.4287930	2.1760778	0.9231228			
8	0.0172267	0.1073447	0.4478257	0.8468243	1.8369024	1.6551557	2.4230264	0.9198113		
9	0.0076767	0.0706048	0.2441854	0.7863109	1.2016071	2.3781188	1.8814798	2.6709468	0.9175476	
10	0.0043067	0.0344971	0.1824512	0.4553892	1.2444914	1.6129856	2.9815094	2.1078524	2.9194657	0.9159

c. 2-dB Ripple (ϵ = 0.7647831 ϵ^2 = 0.5848932)

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	
1	1.3075603									
2	0.6367681	0.8038164								
3	0.3268901	1.0221903	0.7378216							
4	0.2057651	0.5167981	1.2564819	0.7162150						
5	0.0817225	0.4593491	0.6934770	1.4995433	0.7064606					
6	0.0514413	0.2102706	0.7714618	0.8670149	1.7458587	0.7012257				
7	0.0204228	0.1660920	0.3825056	1.1444390	1.0392203	1.9935272	0.6978929			
8	0.0128603	0.0729373	0.3587043	0.5982214	1.5795807	1.2117121	2.2422529	0.6960646		
9	0.0051076	0.0543756	0.1684473	0.6444677	0.8568648	2.0767479	1.3837464	2.4912897	0.6946793	
10	0.0032151	0.0233347	0.1440057	0.3177560	1.0389104	1.1585287	2.6362507	1.5557424	2.7406032	0.69

d. 3-dB Ripple (ϵ = 0.9976283 ϵ^2 = 0.9952623)

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	
1	1.0023773									
2	0.7079478	0.6448996								
3	0.2505943	0.9283480	0.5972404							
4	0.1769869	0.4047679	1.1691176	0.5815799						
5	0.0626391	0.4079421	0.5488626	1.4149847	0.5744296					
6	0.0442467	0.1634299	0.6990977	0.6906098	1.6628481	0.5706979				
7	0.0156621	0.1461530	0.3000167	1.0518448	0.8314411	1.9115507	0.5684201			
8	0.0110617	0.0564813	0.3207646	0.4718990	1.4666990	0.9719473	2.1607148	0.5669476		
9	0.0039154	0.0475900	0.1313851	0.5834984	0.6789075	1.9438443	1.1122863	2.4101346	0.5659234	
10	0.0027654	0.0180313	0.1277560	0.2492043	0.9499208	0.9210659	2.4834205	1.2526467	2.6597378	0.565

Source: Weinberg, Louis. *Network Analysis and Synthesis*, McGraw-Hill, New York, 1962. Reprinted with permission of the author.

TABLE 3.5 ZEROS OF POLYNOMIAL $V_n(s)$ DERIVED FROM THE CHEBYSHEV APPROXIMATION FOR ½-, 1-, 2-, and 3-dB RIPPLES

Chebyshev filter $H_n(s) = \frac{K_n}{V_n(s)}$ $K_n = \begin{cases} b_0 & \text{for } n \text{ odd} \\ b_0/(1 + \epsilon^2)^{1/2} & \text{for } n \text{ even} \end{cases}$

a. 1/2-dB Ripple (ϵ = 0.3493114, ϵ^2 = 0.1220184)

n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n =
−2.8627752	−0.7128122 ±j1.0040425	−0.6264565	−0.1753531 ±j1.0162529	−0.3623196	−0.0776501 ±j1.0084608	−0.2561700	−0.0436201 ±j1.0050021	−0.1984053	−0. ±j1.
		−0.3132282 ±j1.0219275	−0.4233398 ±j0.4209457	−0.1119629 ±j1.0115574	−0.2121440 ±j0.7382446	−0.0570032 ±j1.0064085	−0.1242195 ±j0.8519996	−0.0344527 ±j1.0040040	−0. ±j0.
				−0.2931227 ±j0.6251768	−0.2897940 ±j0.2702162	−0.1597194 ±j0.8070770	−0.1859076 ±j0.5692879	−0.0992026 ±j0.8829063	−0. ±j0.
						−0.2308012 ±j0.4478939	−0.2192929 ±j0.1999073	−0.1519873 ±j0.6553170	−0. ±j0.
								−0.1864400 ±j0.3486869	−0. ±j0.

b. 1-dB Ripple (ϵ = 0.5088471, ϵ^2 = 0.2589254)

n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n =
−1.9652267	−0.5488672 ±j0.8951286	−0.4941706	−0.1395360 ±j0.98333792	−0.2894933	−0.0621810 ±j0.9934115	−0.2054141	−0.0350082 ±j0.9964513	−0.1593305	−0. ±j0.
		−0.2470853 ±j0.9659987	−0.3368697 ±j0.4073290	−0.0894584 ±j0.9901071	−0.1698817 ±j0.7272275	−0.0457089 ±j0.9952839	−0.0996950 ±j0.8447506	−0.0276674 ±j0.9972297	−0 ±j0
				−0.2342050 ±j0.6119198	−0.2320627 ±j0.2661837	−0.1280736 ±j0.7981557	−0.1492041 ±j0.5644443	−0.0796652 ±j0.8769490	−0 ±j0
						−0.1850717 ±j0.4429430	−0.1759983 ±j0.1982065	−0.1220542 ±j0.6508954	−0 ±j0
								−0.1497217 ±j0.3463342	−0 ±j0

c. 2-dB Ripple (ϵ = 0.7647831, ϵ^2 = 0.5848932)

n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n
−1.3075603	−0.4019082 ±0.6893750	−0.3689108	−0.1048872 ±j0.9579530	−0.2183083	−0.0469732 ±j0.9817052	−0.1552958	−0.0264924 ±j0.9897870	−0.1206298	− ±j
		−0.1844554 ±j0.9230771	−0.2532202 ±j0.3967971	−0.0674610 ±j0.9734557	−0.1283332 ±j0.7186581	−0.0345566 ±j0.9866139	−0.0754439 ±j0.8391009	−0.0209471 ±j0.9919471	− ±j
				−0.1766151 ±j0.6016287	−0.1753064 ±j0.2630471	−0.0968253 ±j0.7912029	−0.1129098 ±j0.5606693	−0.0603149 ±j0.8723036	− ±j
						−0.1399167 ±j0.4390845	−0.1331862 ±j0.1968809	−0.0924078 ±j0.6474475	− ±j
								−0.1133549 ±j0.3444996	−0 ±j0

d. 3-dB Ripple (ϵ = 0.9976283, ϵ^2 = 0.9952623)

n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n
1.0023773	−0.3224498 ±j0.7771576	−0.2986202	−0.0551704 ±j0.9464844	−0.1775085	−0.0382295 ±j0.9764060	−0.1264854	−0.0215782 ±j0.9867664	−0.0982716	−0 ±j0
		−0.1493101 ±j0.9038144	−0.2056195 ±j0.3920467	−0.0548531 ±j0.9659238	−0.1044450 ±j0.7147788	−0.0281456 ±j0.9826957	−0.0614494 ±j0.8365401	−0.0170647 ±j0.9895516	−0 ±j0.
				−0.1436074 ±j0.5969738	−0.1426745 ±j0.2616272	−0.0788623 ±j0.7880608	−0.0919655 ±j0.5589582	−0.0491358 ±j0.8701971	−0. ±j0.
						−0.1139594 ±j0.4373407	−0.1084807 ±j0.1962800	−0.0752804 ±j0.6458839	−0. ±j0.
								−0.0923451 ±j0.3436677	−0. ±j0.

Source: Weinberg, Louis. *Network Analysis and Synthesis*, McGraw-Hill, New York, 1962. Reprinted by permission of author.

3.IIR Filter Design by Impulse Invariant Method

$$H(s) \xleftrightarrow{Inv.Laplace} h(t) \xleftrightarrow{t \to nT} h(n) \xleftrightarrow{Z-Transform} H(z)$$

Example

Find the H(z) corresponding to the impulse invariant design a sample rate of 1/T samples/sec. for an analog filter H_a**(s) specified as follows:**

$$H_a(s) = \frac{A}{(s+\alpha)}$$

Solution:

The analog system's impulse invariant response is obtained by taking the inverse Laplace transform of H_a**(s) to give h_a(t) as**

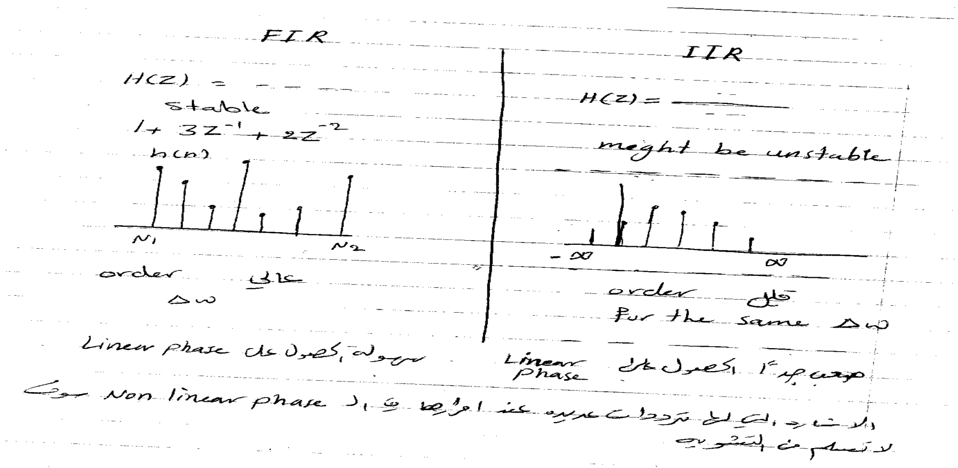
$$h_a(t) = Ae^{-\alpha t}u(t)$$

The corresponding h(n) is then given by

$$h(n) = Ae^{-\alpha nT}u(nT) = A(e^{-\alpha T})^n u(n)$$

And therefore the discrete-time filter has the following Z-transform

$$H(z) = Z[h(n)] = Z[A(e^{-\alpha T})^n u(n)] = \frac{Az}{z - e^{-\alpha T}}$$



Linear phase يمكن الحصول عليه

Linear phase صعب جداً الحصول عليه

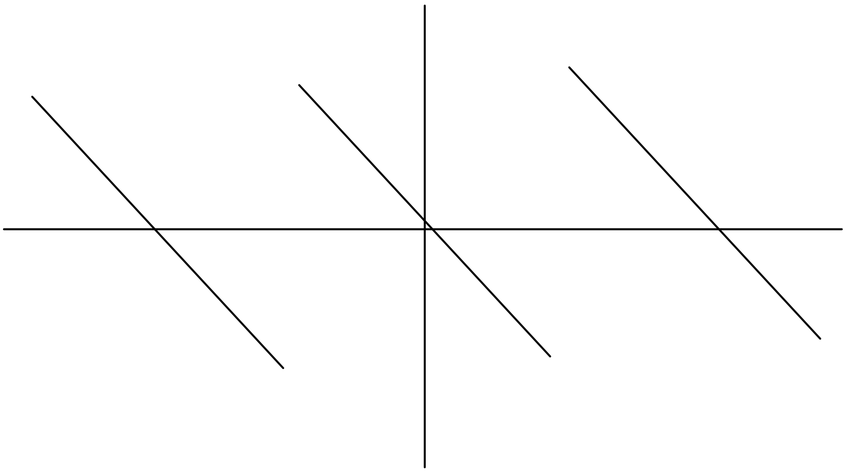
الاشارة التي لها ترددات عديدة عند امرارها في ال Non linear phase سوف
لا تسلم من التشويه

FIR Filter Design

In the previous sections, digital filters were designed to give a desired frequency response magnitude without regard to the phase response. In many cases a linear phase characteristic is required throughout the pass-band of the filter to preserve the shape of a given signal within the pass-band.
Assume a filter with frequency response

$$H(e^{jw}) = \left|H(e^{jw})\right| \; e^{j\varphi(e^{jw})}$$

$$\varphi(e^{jw}) = \alpha w \qquad \rightarrow liner\ phase\ (-\alpha)$$



Why linear phase filter ?

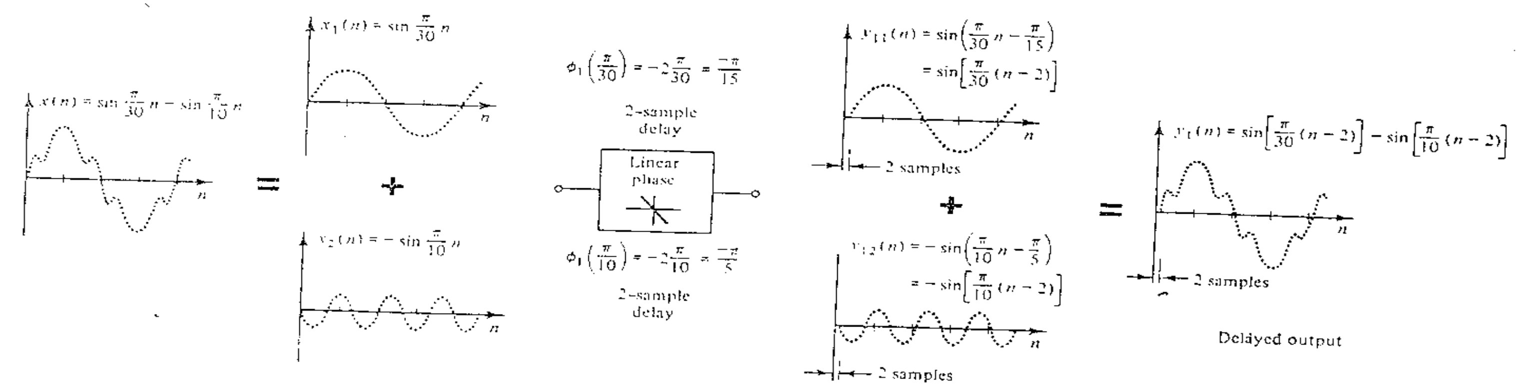
The **linear phase filter did not alter the shape of original signal, simply translated it by an amount α .**
If the phase response had not been liner , the output signal would have been a distorted version .

In Fig.4.11 the responses of two different filters to the same input(a sum of two sinusoidal signals) is presented. The filters have the same magnitude frequency response but differ in their phases as one has linear and the other a quadratic phase. For the filter with liner phase, the sinusoidal components

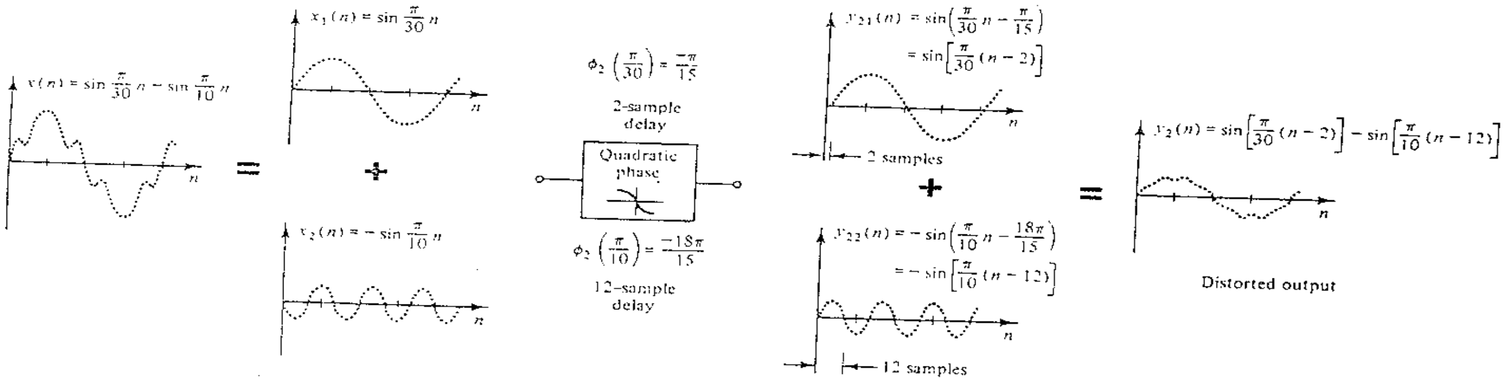
each go through a steady state phase change, but in such a way that the output signal is just a delayed version of the input while the quadratic phase filter causes phase shifts in the two sinusoidal signals resulting in an output that is a distorted version of the input signal.

It can be shown that a casual IIR filter cannot produce a linear phase characteristic and that only special forms of casual FIR filters can give linear phase. This result is clarified in the following theorem.

Theorem : If h(n) represents the impulse response of a discrete-time system, a necessary and sufficient condition for linear phase is that h(n) have finite duration N, and that it by symmetric about its midpoint.



(a) *Linear phase* $\phi_1(\omega) = -2\omega$



(b) *Quadratic phase* $\phi_2(\omega) = \frac{-150}{\pi}\omega^2 + 3\omega$

Figure 4.11 Illustrations of the effect of (a) linear phase and (b) nonlinear phase characteristics on steady state outputs with identical magnitude frequency response curves.

THE DESIGN Concept:-

For a casual FIR filter whose impulse response begins at zero ends at (N-1) , h(n) must satisfy the following:

h(0)=h(N-1) &

h(n)=h(N-1-n) ……for n=0,1,2,……,N-1

for this condition the general shapes of h(n) that give liner phase.

$$H(e^{jw}) = \sum_{-\infty}^{\infty} h(n)\ \ e^{-jwn}$$

$$H(e^{jw}) = \sum_{0}^{N-1} h(n)\ \ e^{-jwn}$$

for N an even number. The summation can be broken into two parts as follows:

$$H(e^{jw}) = \sum_{n=0}^{(N/2)-1} h(n)\ \ e^{-jwn} + \sum_{n=N/2}^{N-1} h(n)\ \ e^{-jwn}$$

Letting m=N-1-n (n=N-1-m) in the second sum gives

$$\sum_{n=N/2}^{N-1} h(n)\ \ e^{-jwn} = \sum_{m=\frac{N}{2}-1}^{0} h(N-1-m)\ \ e^{-jw(N-1-m)}$$

But $h(N-1-m) = h(m)$, and the summation can be reversed to give

$$H(e^{jw}) = \sum_{n=0}^{(\frac{N}{2})-1} h(n)\ \ e^{-jwn} + \sum_{m=0}^{(\frac{N}{2})-1} h(m)\ \ e^{-jw(N-1-m)}$$

Combining yields

$$H(e^{jw}) = \sum_{n=0}^{(\frac{N}{2})-1} h(n)\ \ e^{-jwn} + \sum_{n=0}^{(\frac{N}{2})-1} h(n)\ \ e^{-jw(N-1-n)}$$

$$H(e^{jw}) = \sum_{n=0}^{(\frac{N}{2})-1} 2h(n)\ \ \frac{[e^{-jwn} + \ e^{-jw(N-1-n)}]}{2}$$

$$H(e^{jw}) = e^{-jw((N-1)/2)} \sum_{n=0}^{(\frac{N}{2})-1} 2h(n)\ \ \frac{[e^{-jw(n-\frac{N-1}{2})} + \ e^{-jw(n-\frac{N-1}{2})}]}{2}$$

By factoring we are able to separate $H(e^{jw})$ into two part as follows:

$$H(e^{jw}) = e^{-jw((N-1)/2)} \sum_{n=0}^{(\frac{N}{2})-1} 2h(n)\ \cos\{w[n-(N-1)/2]\} \ldots .N\ even$$

Linear phase   Magnitude

$$\varphi(e^{jw}) = -w\left(\frac{N-1}{2}\right) \qquad \rightarrow N\ even$$

There for, if the sum remains positive, $H(e^{jw})$ has a linear phase with slope $-\left(\frac{N-1}{2}\right)$, for N an odd number , a similar derivation leads to

$$H(e^{jw}) = e^{-jw((N-1)/2)} \left\{ h\left(\frac{N-1}{2}\right) \ + \sum_{n=0}^{(N-3)?2} 2h(n)\ \cos\{w[n-(N-1)/2]\} \right\}$$

DESIGN of FIR filters using windows:

The easiest way to obtain an FIR filter is to simply truncate the impulse response of an IIR filter . If $h_d(n)$ represents the impulse response of a desired IIR filter, then an FIR filter with impulse response h(n) can be obtained as follows:

$$h(n) = \begin{cases} h_d(n) & N1 \le n \le N2 \\ 0, & Otherwise \end{cases}$$

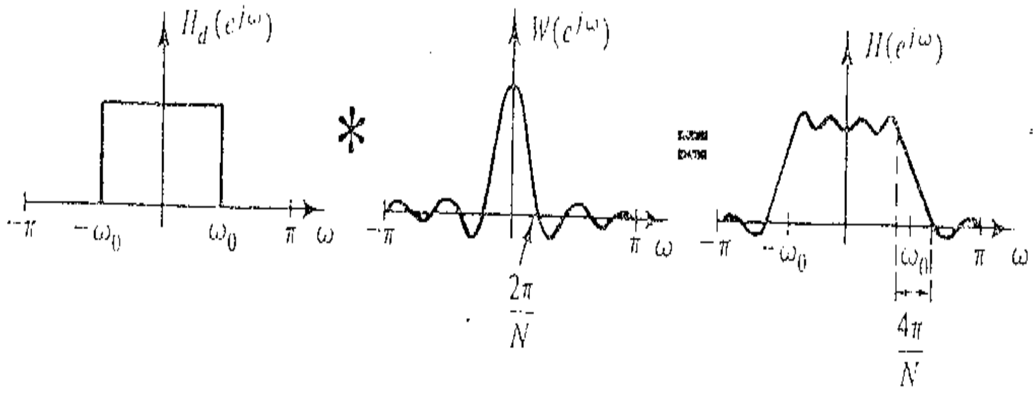
In general, $h(n)$ can be thought of as being formed by the product of $h_d(n)$ and a " Window function" $w(n)$, as follows:

$$h(n) = \quad h_d(n)\, w(n)$$

The frequency response of the resulting filter is the convolution of

$$H(e^{jw}) = \quad H_d(e^{jw}) * W(e^{jw})$$

For example, if $H_d(e^{jw})$ represents an ideal low-pass filter with cutoff frequency ω_o and $w(n)$ is a rectangular window positional about the origin, the $H(e^{jw})$ is shown blow



DESIGN PROCEDURE:-

An ideal low-pass filter with linear phase of slope $-\alpha$ and cutoff ω_c can be characterized in the frequency domain by

$$H_d(e^{j\omega}) = \begin{cases} e^{j\omega\propto} & |\omega| \leq \omega_c \\ 0, & |\omega_c| < |\omega| < \pi \end{cases}$$

Taking the inverse Fourier transform

$$h_d\,(n) = \frac{sin[\omega_c(n-\propto)]}{\pi(n-\alpha)}$$

A casual FIR filter with impulse response $h(n)$ can be obtained by multiplying $h_d\,(n)$ by a window beginning at the origin and ending at N-1 as follows:

$$h(n) = \frac{sin[\omega_c(n-\propto)]}{\pi(n-\alpha)} w(n)$$

For h(n) to be a linear phase filter, α must be selected so that the resulting h(n) is symmetric . As $\frac{sin[\omega_c(n-\propto)]}{\pi(n-\alpha)}$ is symmetric about n= α and the window symmetric about n=(N-1)/2 , a linear phase filter results if the product is symmetric . This requires that

$$\alpha = (N-1)/2$$

Some of the most commonly used windows are the rectangular, Bartlett, Hanning, Hamming, Blackman, and Kaiser windows. These are defined mathematically as follows:

Rectangular: $$w_R\,(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases}$$

Bartlett: $$w_B\,(n) = \begin{cases} \frac{2n}{N-1}, & 0 \le n \le (N-1)/2 \\ \frac{2-2n}{N-1}, & (N-1)/2 \le n \le (N-1) \\ 0 & elsewhere \end{cases}$$

Hanning : $$w_{Han}\,(n) = \begin{cases} \frac{\left\{1-cos\left[\frac{2\pi n}{N-1}\right]\right\}}{2}, & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases}$$

Hamming:

$$w_{Ham}(n) = \begin{cases} 0.54 - 0.46cos\left[\frac{2\pi n}{N-1}\right] & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases}$$

Blackman:

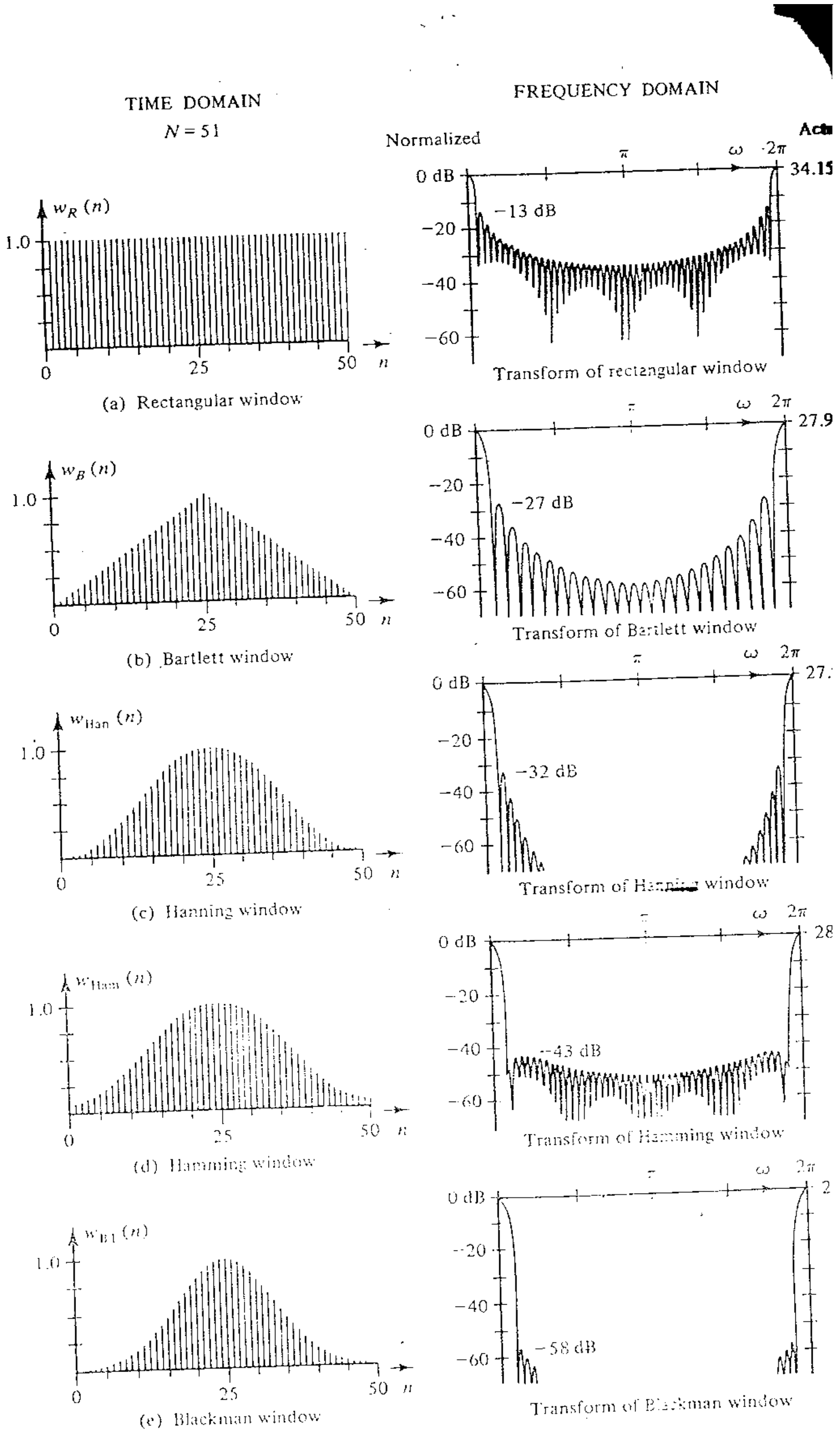
$$w_{Bl}(n) = \begin{cases} 0.42 - 0.5cos\left[\frac{2\pi n}{N-1}\right] + 0.08cos\left[\frac{4\pi n}{N-1}\right], & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases}$$

Kaiser: $$w_K(n) = \begin{cases} \frac{I_o\{w_a\left[\left(\frac{N-1}{2}\right)^2 - \left(n-\frac{N-1}{2}\right)^2\right]^{\frac{1}{2}}\}}{I_o[w_a\left(\frac{N-1}{2}\right)]} & 0 \le n \le N-1 \\ 0 & elsewhere \end{cases}$$

Where $I_o(x)$ is the modified zero order Bessel function of the first kind given by

$$I_o(x) = \int_0^{2\pi} \exp(x \cos\theta)\, d\theta/(2\pi)$$

Plots of the windows and their Fourier transform magnitudes (in decibels) are shown in Fig 4.14 for N=51.



(a) Rectangular window

Transform of rectangular window

(b) Bartlett window

Transform of Bartlett window

(c) Hanning window

Transform of Hanning window

(d) Hamming window

Transform of Hamming window

(e) Blackman window

Transform of Blackman window

Figure 4.14 Plots of windows in the time domain and 20 log magnitude of their Fc tranforms in the frequency domain. Reproduced with modifications from Harris, Fre J., "On the Use of Windows for Harmonic Analysis with the Discrete Fourier T form," Proceedings of the IEEE, Vol. 66, No. 1, Jan 1978.

Design table for FIR low-pass filter design

Window type	**Transition width**	**Minimum stop-band attenuation**	**K**
Rectangular	$4\pi/N$	**-21dB**	**2**
Bartlett	$8\pi/N$	**-25dB**	**4**
Hanning	$8\pi/N$	**-44dB**	**4**
Hamming	$8\pi/N$	**-53dB**	**4**
Blackman	$12\pi/N$	**-74dB**	**6**
Kaiser	**Variable**		

This table although a crude approximation may be used to design a FIR LPF from (k_1, k_2, w_1,w_2) , the following example describe this technique :

Example :- Design a low-pass digital filter to be used in an { A/D- H(z)- D/A} structure that will have a -3dB cutoff of 30π rad./sec and an attenuation of 50dB at 45π rad./sec. The filter is required to have linear phase and the system will use a sampling rate of 100 samples/sec.

Solution:-

The digital specifications obtained are as follows:

$$\omega_1 = \Omega_1\, T = 30\pi\,(0.01) = 0.3\,\pi\, rad, \qquad k_1 = -3dB$$

$$\omega_2 = \Omega_2\, T = 45\pi\,(0.01) = 0.45\,\pi\, rad, \qquad k_2 = -50dB$$

Step 1- **To obtain a stop-band attenuation of – 50dB or more,(from the above table) a Hamming, Blackman, or Kaiser window could be used. The Hamming window is chosen (k=4)**

Step 2- **The approximate number of points needed to satisfy the transition band requirement can be found for**

$$N \geq k \times 2\pi(\omega_2 - \omega_1) = k \times 2\pi(0.45\pi - 0.30\pi) = 53.3$$

To obtain an integer delay the next odd number (N=55) is selected.

Step 3- **select the liner phase of slope $\propto$ and cutoff**

$$\omega_c = \omega_1 = 0.3\pi \qquad rad$$

$$\propto = \frac{N-1}{2} = 27$$

Thus giving a trial impulse response for a window as

$$h(n) = \frac{sin[\omega_c(n - \propto)]}{\pi(n-\alpha)} w(n)$$

For Hamming window $w(n) = w_{Ham} = 0.54 - 0.46cos\left[\frac{2\pi n}{N-1}\right] 0 \leq n \leq N-1$

$$h(n) = \frac{sin\left[\omega_c(n - \frac{N-1}{2})\right]}{\pi\left(n - \frac{N-1}{2}\right)} \{0.54 - 0.46cos\left[\frac{2\pi n}{N-1}\right]\}$$

$$h(n) = \frac{sin[0.3\pi(n-27)]}{\pi(n-27)} \left\{0.54 - 0.46cos\left[\frac{2\pi n}{54}\right]\right\} \qquad 0 \leq n \leq 54$$

Example (Chebyshev HPF)

Design a *Chebyshev* digital filter representing an analog one with the following specifications:

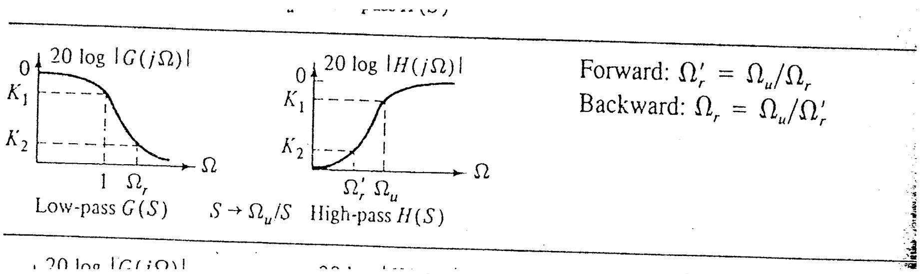
(a) Pass all signals of bilinear warpped frequencies greater than 200rad/sec with no more than 2dB of attenuation.

(b) Stop band attenuation of greater than -20 dB at warpped d frequencies less than 100rad/sec.

Solution:-

The general approach is to first change the requirements to those of a High-pass unit bandwidth PROTOTYPE, design such a LPF , and then apply a LP to HP Transformation to that PROTOTYPE.

From table (3.2) we see



Forward: $\Omega_r' = \Omega_u/\Omega_r$
Backward: $\Omega_r = \Omega_u/\Omega_r'$

$$\Omega_u = 200\frac{rad}{sec}, \quad \Omega_r' = 100\frac{rad}{sec}$$

$$k_1 = 20log\,\frac{1}{\sqrt{1+\epsilon^2}} = -2dB \qquad k_2 = -20dB,$$

The Backward equation then gives the Ω_r for a normalized Low pass prototype.

$\Omega_r = \Omega_u/\Omega_r'$**=200/100=2 rad. /sec**

***The low-pass filter* now has the following specifications**

$$\Omega_1 = 1 \ , \qquad k_1 = 20log\,\frac{1}{\sqrt{1+\epsilon^2}} = -2dB$$

$$\Omega_2 = \ \Omega_r = 2 \ , \qquad k_2 = 20log\left(\frac{1}{A}\right) = -20dB$$

The Chebyshev filter **of order n can then be easily calculated from *the following equations :***

$$n = \left\lceil \frac{log_{10}\left[\left(g+\sqrt{g^2-1}\right)\right]}{log_{10}\left(\Omega_r+\sqrt{\Omega_r{}^2+1}\right)} \right\rceil \text{ where } g = \sqrt{\frac{A^2-1}{\epsilon^2}},\ A = \frac{1}{|H(j\Omega|}$$

$$k_1 = 20log\ \frac{1}{\sqrt{1+\epsilon^2}} = -2dB, \qquad \epsilon^2 = 0.58489$$

$$k_2 = 20log\left(\frac{1}{A}\right) = -20dB, \qquad A = 10$$

$$g = \sqrt{\frac{A^2-1}{\epsilon^2}} = 13$$

$$n = \left\lceil \frac{log_{10}\left[\left(g+\sqrt{g^2-1}\right)\right]}{log_{10}\left(\Omega_r+\sqrt{\Omega_r{}^2+1}\right)} \right\rceil = \left\lceil \frac{log_{10}[(13+\sqrt{13^2-1})]}{log_{10}\left(2+\sqrt{2^2+1}\right)} \right\rceil = 2.47 = 3$$

Using the 2dB ripple part of Table 3.4

$$H_n(s) = \frac{k_n}{V_n(s)}, \qquad k_n = \begin{cases} \dfrac{b_0}{(1+\epsilon^2)^{\frac{1}{2}}} for \quad n \quad even \\ b_0\ for\ n\ odd \end{cases}$$

$$V_n(s) = s^n + b_{n-1}s^{n-1} + \cdots + b_1 s^1 + b_0$$

for n=3, and the fact that <u>since n is odd</u>, we have the desired Chebyshev unit bandwidth low-pass filter as

$$H_3(s) = \frac{b_0}{s^3 + b_2 s^2 + b_1 s + b_0}$$

$$H_{HP}(s) = H_3(s)|_{s\to\ \Omega_u/s} = H_3(s)|_{s\to 200/s}$$

$$H_{HP}(z) = H_{Hp}(s)\Big|_{s\to 2(\frac{1-z}{1+z})}$$